Intersection Graphs and Perfect Graphs
Part I

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Overview

Intersection Graphs

Chordal Graphs

Interval graphs

Unit interval graphs

Comparability graphs and acyclic local tournaments
Basice definitions

- A graph $G$ is a triple consisting of a vertex set $V(G)$, an edge set $E(G)$, and a relation that associates with each edge two vertices (not necessary distinct) called its endpoints.
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- A **loop** is an edge whose endpoints are equal. **Multiple edges** are edges having the same pair of endpoints.
- A **simple graph** is a graph having no loops or multiple edges.
- When two vertices $u$ and $v$ are endpoints of an edge, they are **adjacent** and are **neighbors**.
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- We denote by $N_G(v)$ the set of neighbors of $v$.
- A clique of a graph is a set of pairwise adjacent vertices.
Definitions

- Let $\mathcal{F} = \{S_1, \ldots, S_n\}$ a family of sets. The intersection graph of $\mathcal{F}$ is the graph having $\mathcal{F}$ as vertex set with $S_i$ adjacent to $S_j$ if and only if $i \neq j$ and $S_i \cap S_j \neq \emptyset$. 

- An edge clique cover of a graph $G$ is any family $\varepsilon = \{Q_1, \ldots, Q_k\}$ of cliques of $G$ that cover all the edges of $G$, i.e., if $uv \in E(G)$ then $u, v \in Q_i$ for some $i = 1, \ldots, k$.

- Given a graph $G$ and $\mathcal{F}(\varepsilon) = \{S_v : v \in V(G)\}$, where $S_v = \{i : v \in Q_i\}$. The intersection graph of $\mathcal{F}(\varepsilon)$ is isomorphic to $G$ and $\mathcal{F}(\varepsilon)$ is called the dual set representation of $G$.

- Since every graph has an edge clique cover $\varepsilon = \{\{u, v\} : uv \in E(G)\}$, all graphs are intersection graph of some family of sets.
Definitions

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- An edge clique cover of a graph \( G \) is any family \( \epsilon = \{Q_1, \ldots, Q_k\} \) of cliques of \( G \) that cover all the edges of \( G \), i.e., if \( uv \in E(G) \) then \( u, v \in Q_i \) for some \( i = 1, \ldots, k \).

- Given a graph \( G \) and \( F(\epsilon) = \{S_v : v \in V(G)\} \), where \( S_v = \{i : v \in Q_i\} \). The intersection graph of \( F(\epsilon) \) is isomorphic to \( G \) and \( F(\epsilon) \) is called the dual set representation of \( G \).

- Since every graph has an edge clique cover \( \epsilon = \{\{u, v\} : uv \in E(G)\} \), all graphs are intersection graph of some family of sets.
Example

\[S_{v_1} = \{1\}, \; S_{v_2} = \{1, 2\}, \; S_{v_3} = \{2\}, \; S_{v_4} = \{2, 3, 4, 5\},\]
\[S_{v_5} = \{3, 4\}, \; S_{v_6} = \{1, 4, 5\}.\]
Dual edge clique cover

Given a graph $G$ with a set representation $\mathcal{F} = \{S_{v_1}, \ldots, S_{v_n}\}$, the set $\varepsilon(\mathcal{F}) = \{Q_x : x \in \bigcup_i S_{v_i}\}$ where $Q_x = \{i : x \in S_{v_i}\}$ is an edge clique cover of $G$ and $\varepsilon(\mathcal{F})$ is called a dual edge clique cover of $G$. 
**Dual edge clique cover**

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If $S_{v_1} = \{1\}$, $S_{v_2} = \{1, 2\}$, $S_{v_3} = \{2\}$, $S_{v_4} = \{2, 3\}$, $S_{v_5} = \{3, 4\}$, $S_{v_6} = \{1, 4, 5\}$, then $Q_1 = \{v_1, v_2, v_6\}$, $Q_2 = \{v_2, v_3, v_4\}$, $Q_3 = \{v_4, v_5\}$, $Q_4 = \{v_5, v_6\}$ and $Q_5 = \{v_4, v_6\}$. 
Intersection number

- The intersection number $i(G)$ is the minimum cardinality of a set $S$ such that $G$ is the intersection graph of a family of subsets of $S$. 

\[ \text{Proof sketch:} \quad \text{Let } G \text{ be a graph } \varepsilon \text{ a clique cover } |\varepsilon| = \theta(G), \text{ then } |\bigcup S \in F(\varepsilon) S| = \theta(G) \text{ and so } i(G) \leq \theta(G). \text{ Conversely, if } G \text{ is a graph with a set representation } F = \{ S_1, ..., S_n \} \text{ with } |\bigcup S_i| = i(G), \text{ then } |\varepsilon(F)| = i(G) \text{ and so } \theta(G) \leq i(G). \]

\[ \Box \]
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For every graph $G$, $i(G) = \theta(G)$.

**Proof sketch:** Let $G$ be a graph $\varepsilon$ a clique cover $|\varepsilon| = \theta(G)$, then $|\bigcup_{S \in \mathcal{F}(\varepsilon)} S| = \theta(G)$ and so $i(G) \leq \theta(G)$. Conversely, if $G$ is a graph with a set representation $\mathcal{F} = \{S_1, \ldots, S_n\}$ with $|\bigcup_{i} S_i| = i(G)$, then $|\varepsilon(\mathcal{F})| = i(G)$ and so $\theta(G) \leq i(G)$. □
Example

\[ Q_1 = \{ v_1, v_2, v_6 \}, \quad Q_2 = \{ v_2, v_3, v_4 \}, \quad Q_3 = \{ v_4, v_5, v_6 \} , \]
\[ \varepsilon = \{ Q_1, Q_2, Q_3 \}, \quad \mathcal{F}(\varepsilon) = \{ \{1\}, \{1, 2\}, \{2\}, \{2, 3\}, \{3\}, \{1, 3\} \}, \]
and \( \theta = i = 3 \).
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and \( \theta = i = 3 \).

Kou, Stock-Meyer, & Wong (1978)

It is NP-complete to determine \( \theta(G) = i(G) \).
A chordal graph is an intersection graph of a family of subtrees in a tree.
Graphs with no induced cycle of length at least 4

Remark
If $G$ is a chordal graph, then $G$ contains no induced cycle with at least four edges.
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If $G$ is a chordal graph, then $G$ contains no induced cycle with at least four edges.

Proof sketch: Let $T = \{T_v\}_{v \in V(G)}$ be a family of subtrees of a tree $G$ such that $uv \in E(G)$ iff $V(T_u) \cap V(T_v) \neq \emptyset$. Suppose towards a contradiction that $G$ contains a cycle of length $k$ with $k \geq 4$. Therefore, there exists a collection of subtrees in $T$, $\{T_i\}_{i=1}^{k}$ such that $T_i \cap T_j \neq \emptyset$ iff $|i - j| = 1$ modulo $k$. Consequently, it can be proved that, $T$ contains a cycle, a contradiction.□
A vertex $v$ of $G$ is a simplicial vertex if its neighborhood $N_G(v)$ is a clique of $G$ (a set of pairwise adjacent vertices).
Simplicial vertex and clique separator

- A vertex $v$ of $G$ is a simplicial vertex if its neighborhood $N_G(v)$ is a clique of $G$ (a set of pairwise adjacent vertices).
- A subset $S$ of $V(G)$ is a vertex separator of two nonadjacent vertices $a$ and $b$ (or an $a, b$-separator) if $a$ and $b$ are in different connected components of $G - S$. 
A vertex $v$ of $G$ is a **simplicial vertex** if its neighborhood $N_G(v)$ is a clique of $G$ (a set of pairwise adjacent vertices).

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A subset $S$ of $G$ is a **minimal clique separator** if no proper subset of $S$ is a minimal separator.
Simplicial vertex and clique separator

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- A subset $S$ of $G$ is a **minimal clique separator** if no proper subset of $S$ is a minimal separator.
- Let $\sigma = [v_1, \ldots, v_n]$ be an ordering of the vertices of $G$. We say that $\sigma$ is a **perfect vertex elimination scheme** if $v_i$ is a simplicial vertex of $G[\{v_i, \ldots, v_n\}]$ for each $i = 1, \ldots, n$. 
Example

The graph on the left is a chordal graph, and the graph on the right is not a chordal graph because \( \{o, p, r, z\} \) induces a 4-cycle. Vertices \( a \) and \( e \) are simplicial vertices. The sets \( \{c, f\} \) and \( \{x, r, o\} \) are minimal separators. The ordering \( \sigma = [e, d, c, f, b, a] \) is a perfect vertex elimination scheme.
Structural characterization

Fulkerson and Gross (1965)

Let $G$ be a graph. The following statements are equivalents:

1. $G$ has no induced cycle of length at least four as induced subgraph.

2. Every minimal vertex separator is a clique.
Structural characterization

Fulkerson and Gross (1965)

Let $G$ be a graph. The following statements are equivalents:

1. $G$ has no induced cycle of length at least four as induced subgraph.

2. Every minimal vertex separator is a clique.

Proof sketch: Suppose that $(x_1, x_2, x_3, \ldots, x_n)$ is an induces cycle of $G$ with $n \geq 4$. Therefore, if $S$ is a $x_1, x_3$-separator, then $x_e \in S$ and $x_i \in S$ for some $4 \leq i \leq n$. Consequently, $S$ is not a clique. Arguing, towards a contradiction, suppose that $S$ is a minimal clique separator of $G$ with $x, y \in S$ two nonadjacent vertices. If $G_1$ and $G_2$ are connected components of $G - S$, then each vertex in $S$ has at least one vertex in $V(G_i)$ for $i = 1, 2$. Consequently, there exist paths $P_1$ and $P_2$ of minimum length in $G_1$ and $G_2$ respectively s.t. $xP_1yP2x$ is an induced cycle of length at least four. □
Minimum clique separator

Dirac (1961)
Every graph with no cycle of length at least four as induced subgraph has a simplicial vertex. Moreover, if $G$ is not a complete graph, then $G$ has at least two nonadjacent simplicial vertices.
Minimum clique separator

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Proof sketch: If $G$ is a complete graph the result follows immediately. Assume that $G$ is not a complete graph, then $G$ has a minimal $a, b$-separator $S$ which is a clique with $G_a$ and $G_b$ connected components of $G - S$ containing $a$ and $b$ respectively. By induction, either $H_a = G[V(G_a) \cup S]$ has two nonadjacent simplicial vertices and so one vertex of $G_a$ is a simplicial vertex of $G$, or $H_a$ is a complete graph and so every vertex of $G_a$ is simplicial vertex of $G$. Analogously, $G_b$ also has a simplicial vertex which is also a simplicial vertex of $G$.□
Example

The set $S$ is a minimal clique separator, and $v_2$ and $v_6$ are two nonadjacent simplicial vertices.
Prefect vertex elimination scheme

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Let $G$ be a graph. The following statements are equivalents:

1. $G$ has no induced cycle of length at least four as induced subgraph.

2. Every minimal vertex separator is a clique.

3. $G$ has a perfect vertex elimination scheme.

Proof sketch:

1. $\iff$ 2. was already proved. If $G$ has no induced cycle with at least four edges, then $G$, by virtue of Dirac's theorem, has at least one simplicial vertex $v_1$. By induction, $G - v_1$ has a PVES $\sigma' = [v_2, \ldots, v_n]$ and so $\sigma = [v_1, v_2, \ldots, v_n]$ is a PVES of $G$. Arguing, towards a contradiction, suppose that $G$ has a cycle $C$ with at least four edges and a perfect a PVES. If $v$ is the vertex of $C$ with the smallest index in $\sigma$, then its two neighbors in $C$ are adjacent, a contradiction. $\square$
Prefect vertex elimination scheme

Fulkerson and Gross (1965)

Let $G$ be a graph. The following statements are equivalents:

1. $G$ has no induced cycle of length at least four as induced subgraph.
2. Every minimal vertex separator is a clique.
3. $G$ has a perfect vertex elimination scheme.

Proof sketch: 1. $\Leftrightarrow$ 2. was already proved. If $G$ has a no induced cycle with at least four edges, then $G$, by virtue of Dirac’s theorem, has at least one simplicial vertex $v_1$. By induction, $G - v$ has a PVES $\sigma' = [v_2, \ldots, v_n]$ and so $\sigma = [v_1, v_2, \ldots, v_n]$ is a PVES of $G$. Arguing, towards a contradiction, suppose that $G$ has a cycle $C$ with at least four edges and a perfect a PVES. If $v$ is the vertex of $C$ with the smallest index in $\sigma$, then its two neighbors in $C$ are adjacent, a contradiction. $\square$
Clique tree

Let $G$ be a graph. The following statements are equivalents.

1. $G$ has no induced cycle with at least four edges.
2. $G$ is the intersection graph of a family of subtrees of a tree.
3. There exists a tree $\mathcal{T} = (\mathcal{K}, \mathcal{E})$ whose vertex set $\mathcal{K}$ is the set of maximal cliques of $G$ such that each induced subgraph $\mathcal{T}[\mathcal{K}_v]$ is connected, where $\mathcal{K}_v$ consists of those maximal cliques which contains $v$. 
Clique tree

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- The tree $T$ is called the **clique tree** of $G$.
- Let $G$ be a graph and $v$ a simplicial vertex of $G$,
  \[U = \{ w : N_G(w) \subseteq N_G[v] \} .\]
- \[Y = N_G(v) \setminus U .\]
Example 1: $Y$ is a maximal clique of $G - U$

Let $v = 1$, $U = \{1\}$ and $Y = \{2, 3, 4\}$
Example 2: $Y$ is not a maximal clique of $G - U$

Let $v = 1$, $U = \{1, 2\}$ and $Y = \{3, 8\}$
Example 3
The clique graph $K(G)$ is the intersection graph of the maximal cliques of $K(G)$.

The weighted clique graph $K^w(G)$ is the clique graph of $G$ with each edge $KQ$ given weight $|K \cap Q|$. 

Gavril (1987) A connected graph $G$ is a chordal graph if and only if some maximum spanning tree of $K^w(G)$ is a clique tree of $G$. Moreover, every maximum spanning tree of $K^w(G)$ is a clique tree of $G$, and every clique tree of $G$ is such a maximum spanning tree.
Weighted clique tree

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Example
Procedure LexBFS($x$)

- **Input:** A graph $G$ and a vertex $x$ of $G$.
- **Output:** An ordering $\sigma$ of the vertices of $G$.

1. $\text{label}(x) \leftarrow |V(G)|$;
2. **for** each vertex $y \in V(G) \setminus \{x\}$ **do** $\text{label}(y) \leftarrow \Lambda$;
3. **for** $i \leftarrow |V(G)|$ **downto** 1 **do**
4. pick an unnumbered vertex $y$ with lexicographically the largest label;
5. $\sigma(y) \leftarrow |V(G)| + 1 - i$ (assign to $y$ number $|V(G)| + 1 - i$);
6. **for** each unnumbered vertex $z \in N_G(y)$ **do** append $i$ to $\text{label}(z)$. 
Example
Example
Example
Properties of lexBFS ordering

Dragan, Nicolai & Brandstädt (1997)

An ordering $\prec$ of the vertices of an arbitrary graph $G$ is a lexBFS ordering if and only if for all vertices $a$, $b$, $c$ of $G$ such that $ac \in E(C')$ and $bc \notin E(G)$, $c \prec b \prec a$ implies the existence of a vertex $d$ such that $d$ is adjacent to $b$ but not to $a$ and $d \prec c$. 

Rose, Tarjan & Lueker (1976)

Let $\sigma$ be a lexBFS ordering of a chordal graph $G$ and let $v$ be an arbitrary vertex of $G$. Let $W$ be the set of vertices of $G$ that occur before $v$ in $\sigma$. Then, $v$ is a simplicial vertex in the induced subgraph $G[\{v\} \cup W]$. 


Properties of lexBFS ordering

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An ordering $\prec$ of the vertices of an arbitrary graph $G$ is a lexBFS ordering if and only if for all vertices $a$, $b$, $c$ of $G$ such that $ac \in E(C)$ and $bc \notin E(G)$, $c \prec b \prec a$ implies the existence of a vertex $d$ such that $d$ is adjacent to $b$ but not to $a$ and $d \prec c$.

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Proof sketch

Arguing towards a contradiction, suppose that \( v = v_0 \) is not a simplicial vertex of \( G[\{v\} \cup W] \). Assume w.o.l.o.g that \( v \) is the vertex with the largest index in \( \sigma \) and so \( W = V(G') \setminus \{v\} \).
Proof sketch

Consequently, there exists two vertices $v_1$ and $v_2$ such that $v_1, v_2 \in \mathcal{N}_G(v)$ and $v_1 v_2 \notin E(G)$. Choose $v_2$ the smallest (respect to the ordering $\sigma$). We know that there exists a vertex $v_3$, such that $v_3$ is adjacent to $v_2$ but not to $v_1$, choose $v_3$ as smaller as possible (with respect to $\sigma$). Besides, since $G$ is chordal, $v_2 v_3 \notin E(G)$.
Proof sketch

Consider the longest decreasing sequence (with respect to $\sigma$) $v_0, v_1, v_2, \ldots, v_m$ such that, $v_0v_i \in E(G)$ for $i = 1, 2$, $v_iv_j \in E(G)$ iff $|i - j| = 2$, and $v_j$ is the smallest vertex (respect to $\sigma$) such that $v_j$ is adjacent to $v_{j-1}$ but not to $v_{j-2}$.
Proof sketch

. We know that there exists a vertex $v_{m+1}$ which is adjacent to $v_{m-1}$ but not to $v_{m-2}$ (chosen as smaller as possible with respect to $\sigma$).
Proof sketch

. Arguing towards a contradiction, suppose that $v_{m+1}$ is adjacent to $v_{m-3}$. Therefore, there exists a vertex $w$ greater than $v_{m+1}$ (and so greater than $v_m$) which is adjacent to $v_{m-2}$ but not to $v_{m-3}$, contradicting that $v_m$ is minimum.
Proof sketch

Therefore, $v_{m+1}$ is not adjacent to $v_{m-3}$. 

\[ m = 7 \]

\[ v_7 \quad v_5 \quad v_3 \quad v_1 \quad v_0 \quad v_2 \quad v_4 \quad v_6 \quad v_8 \]
Finally, since $G$ is chordal, $v_{m+1}$ is not adjacent to $v_i$ for each $i = 0, \ldots, m - 4, m$, contradicting the maximality of $m$. □
Chordal graph linear-time recognition algorithm

- **Input:** An arbitrary graph $G$.
- **Output:** A statement declaring whether or not $G$ is a chordal graph.

1. Do an arbitrary lexBFS.
2. If the reverse of the lexBFS ordering $\sigma$ is a perfect vertex elimination ordering, then conclude that $G$ is a chordal graph; else, conclude that $G$ is not a chordal graph.
Gaussian elimination

- Gaussian elimination on an $n \times n$ matrix $M$ consists in choosing a nonzero pivot $m_{ij}$, then using elementary row and column operations to change $m_{ij}$ into 1, and to change $m_{rj}$ and $m_{is}$ into 0 for all $r \neq i$ and $s \neq j$. 

- An elimination scheme is a sequence of $n$ pivots to reduce a nonsingular matrix $M$ entries to the identity matrix.

- A perfect elimination scheme has the further property that no zero entry is ever made nonzero along the way.

- Given a symmetric matrix $M$, the graph of $M$, denoted by $G(M)$, has a vertex set $\{1, 2, \ldots, n\}$ such that $i$ is adjacent to $j$ iff $m_{ij} \neq 0$. 


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An elimination scheme is a sequence of $n$ pivots to reduce a nonsingular matrix $M$ entries to the indentity matrix.

A perfect elimination scheme has the further property that non zero entry is ever made nonzero along the way.

Given a symmetric matrix $M$ the graph of $M$, denoted by $G(M)$, has a vertex set $\{1, 2, \ldots, n\}$ such that $i$ is adjacent to $j$ iff $m_{ij} \neq 0$. 
Example

Bad choice:

\[
\begin{pmatrix}
4 & 1 & 1 \\
1 & 1 & 0 \\
1 & 0 & 1
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 0 & 0 \\
0 & \textbf{3} & -1 \\
0 & -1 & 3
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 8
\end{pmatrix}
\]

Good choice:

\[
\begin{pmatrix}
4 & 1 & 1 \\
1 & 1 & 0 \\
1 & 0 & 1
\end{pmatrix}
\rightarrow
\begin{pmatrix}
4 & 1 & 0 \\
1 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\rightarrow
\begin{pmatrix}
4 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]
A symmetric matrix $M$ with nonzero diagonal entries has a perfect elimination scheme if and only if $G(M)$ is chordal.

Proof sketch:
Pivoting $m_{ii}$ results in removing all the edges incident to the vertex $i$ and simultaneously adding all edges $rs$ whenever $m_{ri} \neq 0$ and $m_{is} \neq 0$. Hence no zero entry is made zero in $M$ precisely when every two neighbors of $i$ are adjacent in $G(M)$; equivalently $i$ is a simplicial vertex of $G(M)$. **Rose (1970)**
A symmetric matrix $M$ with nonzero diagonal entries has a perfect elimination scheme if and only if $G(M)$ is chordal.

**Proof sketch:** Pivoting $m_{ii}$ results in removing all the edges incidents to the vertex $i$ and simultaneously adding all edges $rs$ whenever $m_{ri} \neq 0$ and $m_{is} \neq 0$.

\[
\begin{pmatrix}
  \vdots & \vdots & \vdots \\
  \cdots & m_{ii} & \cdots & m_{is} & \cdots \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  \cdots & m_{ri} & \cdots & 0 & \ddots \\
  \vdots & \vdots & \cdots & \vdots & \vdots \\
\end{pmatrix}
\]

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---

**Rose (1970)**
Definition

- An interval graph $G$ is an intersection graph of a family $\mathcal{F}$ of closed (or open) intervals in the real line.
- The family $\mathcal{F}$ is called an interval model of $G$
Application to scheduling

Consider a collection $C = \{c_i\}$ of courses. Let $T_i$ be the time interval during which course $c_i$ is to take place. Which is the minimum number of classrooms needed to be assigned so that there is no two courses $c_i$ and $c_j$ in the same classroom such that $T_i \cap T_j \neq \emptyset$. 

\[\text{The problem can be solved by proper coloring the interval graph } G, \text{ with } V(G) = \{c_i\} \text{ and } E(G) = \{c_ic_j : T_i \cap T_j \neq \emptyset\}.\]
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The problem can be solved by proper coloring the interval graph $G$, with $V(G) = \{c_i\}_i$ and $E(G) = \{c_i c_j : T_i \cap T_j \neq \emptyset\}$. 

![Diagram of interval graph and coloring]
Coloring an interval graph

- We denote by $\omega(G)$ and $\chi(G)$ to the size of a maximum clique of a graph $G$ and the minimum numbers of colors needed to proper coloring a graph $G$. 

Remark: If $G$ is an interval graph, then $\chi(G) = \omega(G)$.

Proof sketch: Order the vertices of $G$ according to the left endpoints of the interval representation. Let $k = \chi(G)$. Assume that $v$ receives the color $k$. Therefore, the left endpoint of its corresponding interval also belongs to other intervals that already have colors from 1 to $k - 1$. Consequently, $\omega(G) \geq k = \chi(G)$ and so $\omega(G) = \chi(G)$. □
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Coloring an interval graph

- We denote by $\omega(G)$ and $\chi(G)$ to the size of a maximum clique of a graph $G$ and the minimum numbers of colors needed to proper coloring a graph $G$.

**Remark**

If $G$ is an interval graph, then $\chi(G) = \omega(G)$.

**Proof sketch:** Order the vertices of $G$ according to the left endpoints of the interval representation. Let $k = \chi(G)$. Assume that $v$ receives the color $k$. Therefore, the left endpoint of its corresponding interval also belongs to other intervals that already have colors from 1 to $k - 1$. Consequently, $\omega(G') \geq k = \chi(G')$ and so $\omega(G') = \chi(G')$. $\square$
Hajós (1958)

Every interval graph is a chordal graph.
Relationship with chordal graphs

Hajós (1958)

Every interval graph is a chordal graph.

**Proof sketch:** Let $\mathcal{F} = \{I_v\}_{v \in V(G)}$ an interval model of $G$. Arguing towards a contradiction, suppose that $G$ has a cycle $(v_1, v_2, \ldots, v_k)$ with $k \geq 4$. Choose, $p_i \in I_{v_i} \cap I_{v_{i+1}}$ for $i = 1, \ldots, k - 1$. Since $I_{v_{i-1}} \cap I_{v_{i+1}} = \emptyset$, $\{p_i\}_{1 \leq i \leq k}$ either is a strictly increasing or a strictly decreasing sequence. Therefore, $I_{v_1} \cap I_{v_k} = \emptyset$ contradicting that $v_1v_k \in E(G)$. $\square$
Comparability graphs

- A digraph \( D = (V, A) \) has a **transitive orientation** if for each arc \((a, b) \in A\) and \((b, c) \in A\) then \((a, c) \in E\).
- A graph \( G = (V, E) \) is called a **comparability graph** if its edges can be oriented so that the resulting digraph \( D = (V, A) \) has a transitive orientation.
Comparability graphs

A digraph $D = (V, A)$ has a transitive orientation if for each arc $(a, b) \in A$ and $(b, c) \in A$ then $(a, c) \in E$.

A graph $G = (V, E)$ is called a comparability graph if its edges can be oriented so that the resulting digraph $D = (V, A)$ has a transitive orientation.

Ghouila-Houri (1962)

If $G$ is an interval graph, then $\overline{G}$ is a comparability graph.
Proof sketch

Let \( \{I_v\}_{v \in V(G)} \) be an interval model of \( G \). Define the orientation \( A \) of the edges of \( \overline{G} \) as follows: \((u, v) \in D\) if and only if \( I_u \) is fully to the left of \( I_v \).
Characterization

Gilmore & Hoffman (1964)

Let $G$ be a graph. The following statements are equivalent:

1. $G$ is an interval graph.

2. $G$ is a chordal graph and its complement $\overline{G}$ is a comparability graph.

3. The maximal cliques of $G$ can be linearly ordered such that, for every vertex $v$, the cliques containing $v$ occur consecutively.
2. ⇒ 3. Let $D = (V, A)$ be a transitive orientation of $\overline{G}$. Given two maximal cliques of $G$, $C_1$ and $C_2$, it can be proved that: i) there exists one arc with one of its endpoints in $C_1$ and the other one in $C_2$; and ii) All such arcs in $D$ connecting $C_1$ and $C_2$ have the same orientation.
Proof sketch

Consider the following relation on the collection $\mathcal{C}$ of maximal cliques of $G$: $C_1 < C_2$ if there exists an arc in $A$ connecting a vertex in $C_1$ with a vertex in $C_2$ oriented from $C_1$ toward $C_2$. It can be proved that $(\mathcal{C}, <)$ is a transitive tournament.
Proof sketch

Arguing towards a contradiction, suppose that there exists a vertex $v$, and $C_i, C_j, C_k \in \mathcal{C}$ such that $C_i < C_j < C_k$ and $v \in C_h$ for $h = i, k$ and $v \notin C_j$. Therefore, there exists a vertex $w \in C_j$ such that $vw \in E(G)$. In addition, $v \in C_i$ implies $(v, w) \in D$ and $x \in C_k$ implies $(w, v) \in D$, a contradiction.

3. $\implies$ 1.. For each vertex $v \in V(G)$ we define $I_v$ as the minimal closed interval in $\mathbb{R}$ containing the set of integers $\{i : v \in C_i\}$. It is easy to see that $vw \in E(G)$ if and only if $I_v \cap I_w \neq \emptyset$. $\square$
Clique matrix & consecutive one property

- The **clique matrix** of a graph $G$ is the maximal cliques versus vertices incident matrix.

- A matrix whose entries are zeros and ones has the **consecutive 1's property for columns** if its rows can be permuted in such a way that the 1's in each column occur consecutively.
Clique matrix & consecutive one property

- The **clique matrix** of a graph $G$ is the maximal cliques versus vertices incident matrix.

- A matrix whose entries are zeros and ones has the **consecutive 1’s property for columns** if its rows can be permuted in such a way that the 1’s in each column occur consecutively.

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 \\
\end{bmatrix}
\cdot
\begin{bmatrix}
1 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & \\
0 & 1 & 0 & 0 & \\
1 & 0 & 1 & 1 & \\
1 & 1 & 0 & 0 & \\
\end{bmatrix}
= 
\begin{bmatrix}
1 & 0 & 0 & 1 & \\
1 & 0 & 1 & 1 & \\
1 & 1 & 1 & 0 & \\
1 & 1 & 0 & 0 & \\
0 & 1 & 0 & 0 & \\
\end{bmatrix}
\]
Consecutive 1’s property for columns

Fulkerson & Gross (1965)

A graph $G$ is an interval graph if and only if its clique matrix $M$ has the consecutive 1’s property for columns
Characterization of Interval Graphs

- Three vertices in a graph $G$ form an **asteroidal triple** if every two of them are connected by a path avoiding the third one.

**Boland and Lekkerkerker (1962)**

Let $G$ be a graph, the following statements are equivalent:

1. The graph $G$ is an interval graph.
2. The graph $G$ is chordal and contains no asteroidal triple.
3. The graph $G$ does not contain any of the following graphs as induced subgraphs.

- Bipartite claw
- $n$-net, $n \geq 2$
- Umbrella
- $n$-tent, $n \geq 3$
- $C_n$, $n \geq 4$
Another characterization

Olariu (1991)

For a graph $G$ the following two statements are equivalent:

1. $G$ is an interval graph.
2. There exists a linear order $<$ on $V(G)$ such that for every choice of vertices $u$, $v$ and $w$ with $u < v$ and $v < w$, $uw \in E(G)$ implies $uv \in E(G)$. 
Olariu (1991)

For a graph $G$ the following two statements are equivalent:

1. $G$ is an interval graph.

2. There exists a linear order $<$ on $V(G)$ such that for every choice of vertices $u$, $v$ and $w$ with $u < v$ and $v < w$, $uw \in E(G)$ implies $uv \in E(G)$.

**Proof Sketch:** 1. $\Rightarrow$ 2. Let $I_u = [u_\ell, u_r]$ and $I_v = [v_\ell, v_r]$. Let $G$ be an interval graph with an interval model $F$ we define a linear ordering on $V(G)$ in such a way that $u_\ell < v_\ell$, or $u_\ell = v_\ell$ and $u_r \leq v_r$, whenever $u < v$. It can be proved that such a linear ordering on $V(G)$ satisfies the condition 2.
Another characterization

Olariu (1991)

For a graph $G$ the following two statements are equivalent:

1. $G$ is an interval graph.

2. There exists a linear order $<$ on $V(G)$ such that for every choice of vertices $u$, $v$ and $w$ with $u < v$ and $v < w$, $uw \in E(G)$ implies $uv \in E(G)$.

Proof Sketch: 2. $\Rightarrow$ 1.. We enumerate the vertices of $G$ as $w_1, \ldots, w_n$ in such a way that $w_i < w_j$ whenever $i < j$. First, we will prove that $G$ is chordal. It can be easily proved that $[w_n, w_{n-1} \ldots, w_2, w_1]$ is a perfect vertex elimination scheme.
Olariu (1991)

For a graph $G$ the following two statements are equivalent:

1. $G$ is an interval graph.

2. There exists a linear order $<$ on $V(G)$ such that for every choice of vertices $u, v$ and $w$ with $u < v$ and $v < w$, $uw \in E(G)$ implies $uv \in E(G')$.

**Proof Sketch:** Finally it remains to prove that $\overline{G}$ is a comparability graph. We define the digraph $D$ in such a way that $V(G) = V(D)$ and $(u, v) \in A(D)$ if and only if $u < v$ and $uv \notin E(G')$. Consequently if $(u, v) \in A(D)$ and $(v, w) \in A(D)$, then $u < w$ and $uw \notin E(G')$ because otherwise $uv \in E(G)$. $\square$
Definitions and characterizations

- A unit interval graph is an interval graph having an interval model $\mathcal{F}$ such that all intervals in $\mathcal{F}$ have the same length. Such an interval model is called a unit interval model of the graph.

- A proper interval graph is an interval graph having an interval model $\mathcal{F}$ with no interval properly contained in another interval. Such an interval model is called a proper interval model of the graph.
Definitions and characterizations

- A **unit interval graph** is an interval graph having an interval model $F$ such that all intervals in $F$ have the same length. Such an interval model is called a **unit interval model** of the graph.

- A **proper interval graph** is an interval graph having an interval model $F$ with no interval properly contained in another interval. Such an interval model is called a **proper interval model** of the graph.

**Roberts (1969)**

Given an interval graph $G$ the following conditions are equivalent:

1. $G$ is a proper interval graph.
2. $G$ is a unit interval graph
3. $G$ is an interval graph contains no induced claw.
Proof sketch [Bogart & West (1999)]

1. and 2. \(\Rightarrow\) 3. In an interval representation of the claw the intervals for the three vertices of degree one must be pairwise disjoint, and so the vertex of degree three contains the interval of degree one in the meddle.
3. ⇒ 1. Let $G$ a claw-free interval graph. Since $G$ is claw-free, there is no pair of vertices $v$ and $w$ such that $I_v = [v_1, v_2]$ is properly contained in $I_w = [w_1, w_2]$ such that there is endpoints in $[w_1, v_1]$ and $[v_2, w_2]$ of intervals that do not intersect $I_v$. Hence we can extend $I_v$ past the end of $I_w$ on one end without changing the graph obtained from the representation. Repeating until no more pairs of intervals are related by inclusion yields a proper interval representation.
Proof sketch [Bogart & West (1999)]

1. ⇒ 2. We process the representation from left to right, adjusting all the intervals to length 1. Let \( I_x = [a, b] \) the interval having the leftmost left endpoint. Let \( \alpha = a \) unless \( I_x \) contains the right endpoint of some other interval, in which case let \( \alpha \) be the largest such right endpoint. Hence \( \alpha < \min\{a + 1, b\} \).
Now, adjusting the portion representation of $[a, +\infty)$ by shrinking or expanding $[\alpha, b]$ to $[\alpha, a + 1]$ and translating $[b, +\infty)$ to $[a + 1, \infty)$. □
Proof sketch [Bogart & West (1999)]

Now, adjusting the portion representation of \([a, +\infty)\) by shrinking or expanding \([\alpha, b]\) to \([\alpha, a + 1]\) and translating \([b, +\infty)\) to \([a + 1, \infty)\). □
Another characterization

Looges & Olariu (1993)

A graph $G$ is a proper interval graph if and only if there exists a linear ordering on $V(G)$ such that for every choice of $u, v, w \in V(G)$, $u < v < w$ and $uw \in E(G)$ implies that $uv \in E(G)$ and $vw \in E(G)$.
Proof sketch

Let $I_u = [u_\ell, u_r]$ and $I_v = [v_\ell, v_r]$. Let $G$ be a proper interval graph with an interval model $\mathcal{F}$ we define a linear ordering on $V(G)$ in such a way that $u_\ell < v_\ell$, or $u_\ell = v_\ell$ and $u_r \leq v_r$, whenever $u < v$. It can be proved that such a linear ordering on $V(G)$ satisfies the condition of the necessary condition of the statement.
Proof sketch

⇐ Since the linear ordering on $V(G')$ satisfies the condition of the linear ordering for interval graph of Olariu’s characterization, in virtue of Robert’s characterization, it suffices to show that $G'$ is a claw-free graph. Arguing towards a contradiction, suppose that \{a, b, c, d\} induces a claw in $G$ being $a$ the vertex of degree three. It is easy to see that $a$ cannot precede (follow) $b, c, d$. We can assume w.o.l.o.g that $b$ precedes $a, c, d$ and $d$ follows $a, b, c$. Either case leads to a contradiction. □
Proof sketch

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Definitions

- Recall that a comparability graph is a graph having a transitive orientations of its edges.

- A quasi-transitive orientation of a graph $G$ is an orientation $A(G)$ of if edges in such a way that if $(u, v) \in A(G)$ and $(v, w) \in A(G)$, then $u, w \in E(G)$.

- Given a graph $G$ we define $G^+$ as the graph whose vertex set consists of all ordered pairs $(u, v)$ with $uv \in E(G)$, a vertex $(u, v)$ is adjacent in $G^+$ to $(v, u)$, to any $(w, u)$ such that $vw \notin E(G)$ and to any $(v, w)$ such that $uw \notin E(G)$.
Quasi-transitive orientations

Ghouilà-Houri (1962)

The edges of a graph $G$ can be quasi-transitively oriented if and only if $G^+$ is a bipartite graph.
Lexicographic-two-coloring of $G^+$

- A graph $G$ whose vertices are $1, 2, \ldots, n$.
- A two-coloring of $G^+$

1. while there is an uncolored vertex in $G^+$ do
2. assign color A to the lexicographic smallest unclored vertex $(u, v) \in G^+$
3. complete to the unique two-coloring the component containing $(u, v)$ assigning the color A to those vertices at even distance from $(u, v)$ and the color B to those vertices at odd distance from $(u, v)$. 
Hell and Huang (1995)

Let $D$ be a quasi-transitive orientation of a quasi-transitive orientable graph $G$ in which $(u, v) \in A(D)$ just if $(u, v)$ obtains the color $A$ in the lexicographic-two-coloring of $G^+$. Then $D$ is a transitive orientation of $G$. 
Comparability graph

Hell and Huang (1995)

Let $D$ be a quasi-transitive orientation of a quasi-transitive orientable graph $G$ in which $(u, v) \in A(D)$ just if $(u, v)$ obtains the color $A$ in the lexicographic-two-coloring of $G^+$. Then $D$ is a transitive orientation of $G$.

Proof sketch: Arguing towards a contradiction suppose that \{u, v, w\} forms the smallest ordered triple in the lexicographic ordering for which the conditions of transitively fails. Suppose, w.o.l.o.g, that $(u, v), (v, w), (w, u) \in A(D)$ and $v > w$. So $(v, w)$ was not the first vertex colored $A$ in its connected component. Suppose that $(v', w')$ was the first and so $v' < w'$ and \{u, v', w'\} forms an smaller ordered triple than \{u, v, w\}, and there is an even length path $(v, w) = (v_0, w_0), (v_1, w_1), \ldots, (v_{2k}, w_{2k}) = (v', w')$ in $G^+$.
Hell and Huang (1995)

Let $D$ be a quasi-transitive orientation of a quasi-transitive orientable graph $G$ in which $(u, v) \in A(D)$ just if $(u, v)$ obtains the color $A$ in the lexicographic-two-coloring of $G^+$. Then $D$ is a transitive orientation of $G$.

It can be proved that for each even $(u, v_i), (v_i, w_i), (w_i, u) \in A(D)$ and for each odd $(u, w_i), (w_i, v_i), (v_i, u) \in A(G)$. Therefore, $(u, v_{2k}), (v_{2k}, w_{2k}), (w_{2k}, u) \in A(G)$, contradicting that is \{u, v, w\} is the minimum triple with this property.□
Corollary

Ghouilà-Houri (1962)

A graph $G$ is a comparability graph if and only if $G$ is quasi-transitively orientable.
Corollary

Ghouilà-Houri (1962)

A graph $G$ is a comparability graph if and only if $G$ is quasi-transitively orientable

$O(m\Delta)$ recognizing algorithm

1. Construct $G^+$. 

2. While there exists uncolored vertices do color by A the lexicographically smallest uncored vertex $(u, v)$ use BFS to two-color (if possible) the component of $G^+$ which contains $(u, v)$

3. If some component could not be two-colored then report that $G$ is not a comparability graph.

4. Orient the edge $uv$ of $G$ as $(u, v)$ just if $(u, v)$ obtained color A.
Local tournament

- A quasi-transitive orientation of a graph $G$ is an orientation $A(G)$ of if edges in such a way that if $(u, v) \in A(G)$ and $(v, w) \in A(G)$, then $u, w \in E(G)$.
- Given a graph $G$ we define $G^*$ as the graph whose vertex set consists of all ordered pairs $(u, v)$ with $uv \in E(G)$, a vertex $(u, v)$ is adjacent in $G^*$ to $(v, u)$, to any $(u, w)$ such that $vw \notin E(G)$ and to any $(w, v)$ such that $uw \notin E(G)$. 

Hell & Hoang (1995) The graph $G$ has a local tournament orientation if and only if $G^*$ is a bipartite graph.
Local tournament

- A quasi-transitive orientation of a graph $G$ is an orientation $A(G)$ of its edges in such a way that if $(u, v) \in A(G)$ and $(v, w) \in A(G)$, then $u, w \in E(G)$.

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Hell & Hoang (1995)

The graph $G$ has a local tournament orientation if and only if $G^*$ is a bipartite graph.
Proper interval graph

Hell & Hoang (1995)
A graph is a proper interval graph if and only if it is orientable as an acyclic local tournament.
Proper interval graph

Hell & Hoang (1995)
A graph is a proper interval graph if and only if it is orientable as an acyclic local tournament.

Proof sketch: ⇒ If \( F \) is a proper interval model of \( G \) an acyclic local tournament is obtained by oriented the edge \( uv \) as \((u, v)\), whenever the interval representing \( u \) contains the left endpoint of the interval representing \( v \).
Proper interval graph

Hell & Hoang (1995)

A graph is a proper interval graph if and only if it is orientable as an acyclic local tournament.

**Proof sketch:** \( \Rightarrow \) If \( \mathcal{F} \) is a proper interval model of \( G \) an acyclic local tournament is obtained by oriented the edge \( uv \) as \((u, v)\), whenever the interval representing \( u \) contains the left endpoint of the interval representing \( v \).

\( \Leftarrow \) If \( D \) is a local tournament orientation of \( G \), the we can enumerate the vertices of \( D \) as \( v_1, \ldots, v_n \) so tat for each \( i \) there exists positive integers \( k, l \) such that the vertex \( v_i \) has inset \( \{v_{i-1}, \ldots, v_{i-k}\} \) and outset \( \{v_{i+1}, \ldots, v_{i+l}\} \). Each vertex \( v_i \) can be represented as the interval \([j, j + d_j^+ + 1 - \frac{d_j^+}{d_j^+ + 1}]\), where \( d_j^+ \) is the outdegree of the vertex \( j \).
Acyclic local tournament

Hoang (1993)

Let $G$ be a proper interval graph and let $1, 2, \ldots, n$ be a vertex perfect elimination scheme. Let $D$ be the orientation of $G$ if in which $(u, v) \in A(D)$ if $(u, v)$ obtains color A in the lexicographic two-coloring of $G^*$. Then $D$ is an acyclic local tournament of $G$. 
Polynomial-time recognition algorithm

1. Rename the vertices of $G$ so that $1, 2, \ldots, n$ is a perfect vertex elimination scheme. If $G$ does not admit such a PVES, then report $G$ is not a proper interval graph.

2. Construct $G^*$.

3. While there exist uncolored vertices do color by A the lexicographically smallest uncolored vertex $(uv)$ use BFS to two-color (if possible) the component of $G^*$ which contains $(u, v)$.

4. If some component could not be two-colored then report $G$ is not a proper interval graph.

5. Orient the edge $uv$ as $(u, v)$ if $(u, v)$ obtained color A.

6. If the resulting oriented graph contains a directed cycle then report that $G$ is not a proper interval graph.

7. Construct a proper interval model.
Thank you for your attention!
Kenneth P. Bogart and Douglas B. West.
A short proof that 'proper = unit'.

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