Intersection Graphs and Perfect Graphs
Part I

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Overview

Basice definitions

- A graph G is a triple consisting of a vertex set $V(G)$, an edge set $E(G)$, and a relation that associates with each edge two vertices (not necessary distinct) called its endpoints.

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- A clique of a graph is a set of pairwise adjacent vertices.

Definitions

- Let $\mathcal{F} = \{S_1, \ldots, S_n\}$ a family of sets. The intersection graph of $\mathcal{F}$ is the graph having $\mathcal{F}$ as vertex set with S_i adjacent to S_j if and only if $i \neq j$ and $S_i \cap S_j \neq \emptyset$.

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- An edge clique cover of a graph G is any family $\varepsilon = \{Q_1, \ldots, Q_k\}$ of cliques of G that cover all the edges of G, i.e, if $uv \in E(G)$ then $u, v \in Q_i$ for some $i = 1, \ldots, k$.

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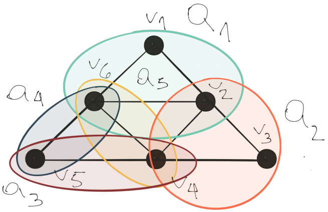
- Since every graph has an edge clique cover $\varepsilon = \{\{u, v\} : \ uv \in E(G)\}$, all graphs are intersection graph of some family of sets.

Example

$S_{v_1} = \{1\}$, $S_{v_2} = \{1, 2\}$, $S_{v_3} = \{2\}$, $S_{v_4} = \{2, 3, 4, 5\}$, $S_{v_5} = \{3, 4\}$, $S_{v_6} = \{1, 4, 5\}$.



Dual edge clique cover

- Given a graph G with a set representation $\mathcal{F} = \{S_{v_1}, \ldots, S_{v_n}\}$, the set $\varepsilon(\mathcal{F}) = \{Q_x : \; x \in \bigcup_i S_{v_i}\}$ where $Q_x = \{i : x \in S_{v_i}\}$ is an edge clique cover of G and $\varepsilon(\mathcal{F})$ is called a dual edge clique cover of G.

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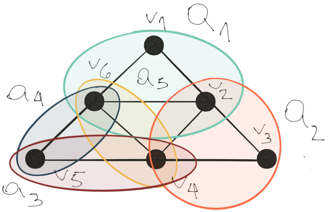
If $S_{v_1} = \{1\}$, $S_{v_2} = \{1,2\}$, $S_{v_3} = \{2\}$, $S_{v_4} = \{2,3\}$, $S_{v_5} = \{3,4\}$, $S_{v_6} = \{1,4,5\}$, then $Q_1 = \{v_1, v_2, v_6\}$, $Q_2 = \{v_2, v_3, v_4\}$, $Q_3 = \{v_4, v_5\}$, $Q_4 = \{v_5, v_6\}$ and $Q_5 = \{v_4, v_6\}$.



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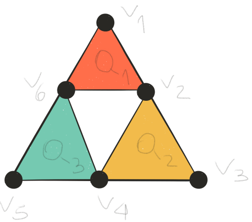
Proof sketch: Let G be a graph ε a clique cover $|\varepsilon| = \theta(G)$, then $|\bigcup_{S \in \mathcal{F}(\varepsilon)} S| = \theta(G)$ and so $i(G) \leq \theta(G)$. Conversely, if G is a graph with a set representation $\mathcal{F} = \{S_1, \ldots, S_n\}$ with $|\bigcup_i S_i| = i(G)$, then $|\varepsilon(\mathcal{F})| = i(G)$ and so $\theta(G) \leq i(G)$.$\square$

Example

$Q_1 = \{v_1, v_2, v_6\}$, $Q_2 = \{v_2, v_3, v_4\}$, $Q_3 = \{v_4, v_5, v_6\}$,
$\varepsilon = \{Q_1, Q_2, Q_3\}$, $\mathcal{F}(\varepsilon) = \{\{1\}, \{1, 2\}, \{2\}, \{2, 3\}, \{3\}, \{1, 3\}\}$,
and $\theta = i = 3$.



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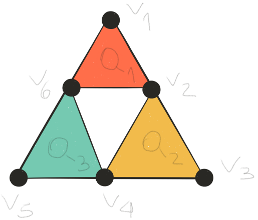
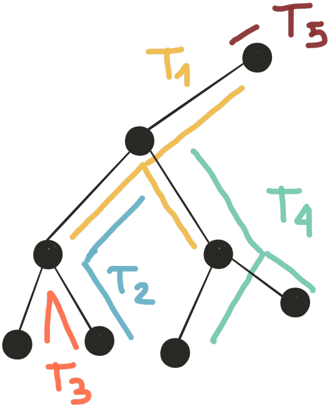


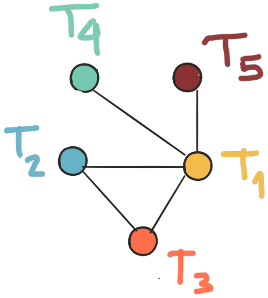
Kou, Stock-Meyer,& Wong (1978)

It is NP-complete to determine $\theta(G) = i(G)$.

Intersection graph of subtrees in a tree

- A chordal graph is an intersection graph of a family of subtrees in a tree.





Graphs with no induced cycle of length at least 4

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Proof sketch: Let $\mathcal{T} = \{T_v\}_{v \in V(G)}$ be a family of subtrees of a tree G such that $uv \in E(G)$ iff $V(T_u) \cap V(T_v) \neq \emptyset$. Suppose towards a contradiction that G contains a cycle of length k with $k \geq 4$. Therefore, there exists a collection of subtrees in $\mathcal{T}$, $\{T_i\}_{i=1}^{k}$ such that $T_i \cap T_j \neq \emptyset$ iff $|i - j| = 1$ modulo k. Consequently, it can be proved that, T contains a cycle, a contradiction.$\square$

Simplicial vertex and clique separator

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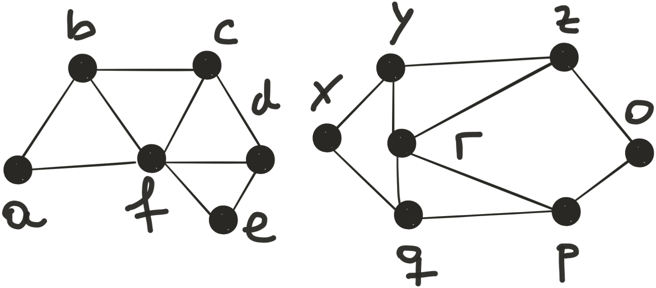
- A subset S of G is a minimal clique separator if no proper subset of S is a minimal separator.
- Let $\sigma = [v_1, \ldots, v_n]$ be an ordering of the vertices of G. We say that σ is a perfect vertex elimination scheme if v_i is a simplicial vertex of $G[\{v_i, \ldots, v_n\}]$ for each $i = 1, \ldots, n$.

Example

The graph on the left is a chordal graph, and the graph on the right is not a chordal graph because $\{o, p, r, z\}$ induces a 4-cycle. Vertices a and e are simplicial vertices. The sets $\{c, f\}$ and $\{x, r, o\}$ are minimal separators. The ordering $\sigma = [e, d, c, f, b, a]$ is a perfect vertex elimination scheme.



Structural characterization

Fulkerson and Gross (1965)

Let G be a graph. The following statements are equivalents:

1. G has no induced cycle of length at least four as induced subgraph.
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Proof sketch: Suppose that $(x_1, x_2, x_3, \ldots, x_n)$ is an induces cycle of G with $n \geq 4$. Therefore, if S is a x_1, x_3-separator, then $x_e \in S$ and $x_i \in S$ for some $4 \leq i \leq n$. Consequently, S is not a clique. Arguing, towards a contradiction, suppose that S is a minimal clique separator of G with $x, y \in S$ two nonadjacent vertices. If G_1 and G_2 are connected components of $G - S$, then each vertex in S has at least one vertex in $V(G_i)$ for $i = 1, 2$. Consequently, there exist paths P_1 and P_2 of minimum length in G_1 and G_2 respectively s.t. xP_1yP2x is an induced cycle of length at least four.$\square$

Minimum clique separator

Dirac (1961)

Every graph with no cycle of length at least four as induced subgraph has a simplicial vertex. Moreover, if G is not a complete graph, then G has at least two nonadjacent simplicial vertices.

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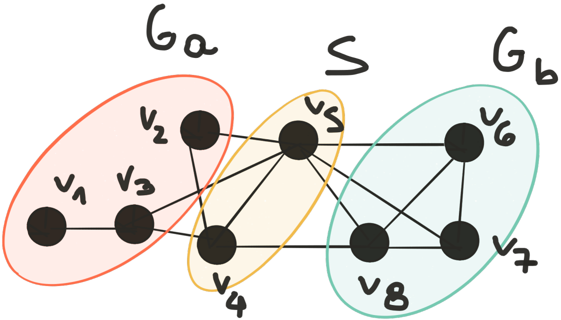
Proof sketch: If G is a complete graph the result follows immediately. Assume that G is not a complete graph, then G has a minimal a, b-separator S which is a clique with G_a and G_b connected components of $G - S$ containing a and b respectively. By induction, either $H_a = G[V(G_a) \cup S]$ has two nonadjacent simplicial vertices and so one vertex of G_a is a simplicial vertex of G, or H_a is a complete graph and so every vertex of G_a is simplicial vertex of G. Analogously, G_b also has a simplicial vertex which is also a simplicial vertex of G.$\square$

Example

The set S is a minimal clique separator, and v_2 and v_6 are two nonadjacent simplicial vertices.



Prefect vertex elimination scheme

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Proof sketch: 1. $\Leftrightarrow$ 2. was already proved. If G has a no induced cycle with at least four edges, then G, by virtue of Dirac's theorem, has at least one simplicial vertex v_1. By induction, $G - v$ has a **PVES** $\sigma' = [v_2, \ldots, v_n]$ and so $\sigma = [v_1, v_2, \ldots, v_n]$ is a **PVES** of G. Arguing, towards a contradiction, suppose that G has a cycle C with at least four edges and a perfect a **PVES**. If v is the vertex of C with the smallest index in σ, then its two neighbors in C are adjacent, a contradiction.$\Box$

Clique tree

Let G be a graph. The following statements are equivalents.

1. G has no induced cycle with at least four edges.
2. G is the intersection graph of a family of subtrees of a tree.
3. There exists a tree $\mathcal{T} = (\mathcal{K}, \mathcal{E})$ whose vertex set $\mathcal{K}$ is the set of maximal cliques of G such that each induced subgraph $T[\mathcal{K}_v]$ is connected, where $\mathcal{K}_v$ consists of those maximal cliques which contains v.

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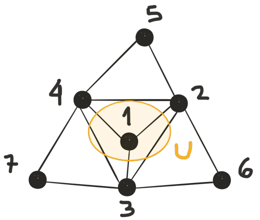
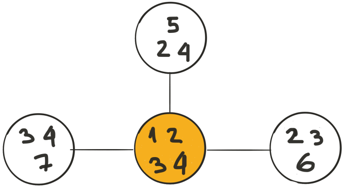
- The tree $\mathcal{T}$ is called the clique tree of G.
- Let G be a graph and v a simplicial vertex of G, $U = \{w : \ N_G(w) \subseteq N_G[v]\}$.
- $Y = N_G(v) \setminus U$.

Example 1: Y is a maximal clique of $G - U$

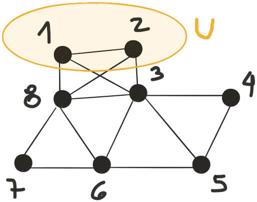
Let $v = 1$, $U = \{1\}$ and $Y = \{2, 3, 4\}$

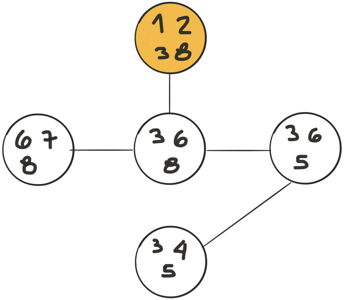




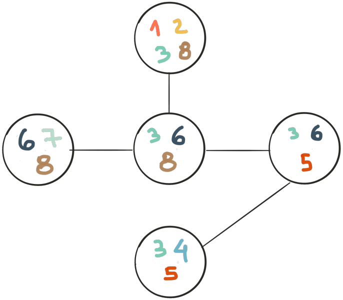
Example 2: Y is not a maximal clique of $G - U$

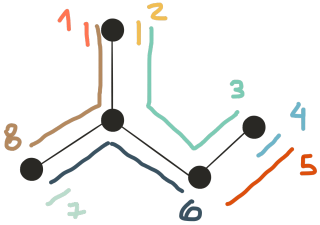
Let $v = 1$, $U = \{1, 2\}$ and $Y = \{3, 8\}$





Example 3





Weighted clique tree

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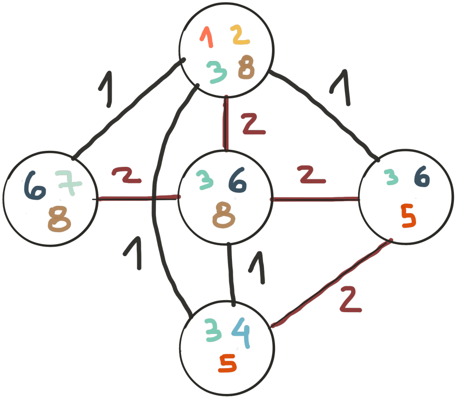
Gavril (1987)

A connected graph G is a chordal graph if and only if some maximum spanning tree of $K^w(G)$ is a clique tree of G. Moreover, every maximum spanning tree of $K^w(G)$ is a clique tree of G, and every clique tree of G is such a maximum spanning tree.

Example

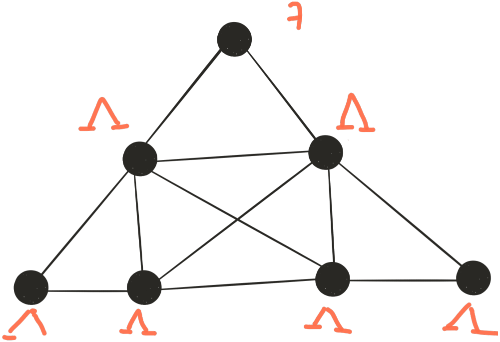


Procedure LexBFS(x)

- **Input:** A graph G and a vertex x of G.
- **Output:** An ordering σ of the vertices of G.

1. label(x)$\leftarrow |V(G)|$;
2. **for** each vertex $y \in V(G) \setminus \{x\}$ **do** label(y)$\leftarrow \Lambda$;
3. **for** $i \leftarrow |V(G)|$ **downto** 1 **do**
4. pick an unnumbered vertex y with lexicographically the largest label;
5. $\sigma(y) \leftarrow |V(G)| + 1 - i$ (assign to y number $|V(G) + 1 - i$);
6. **for** each unnumbered vertex $z \in N_G(y)$ **do** append i to label(z).

Example



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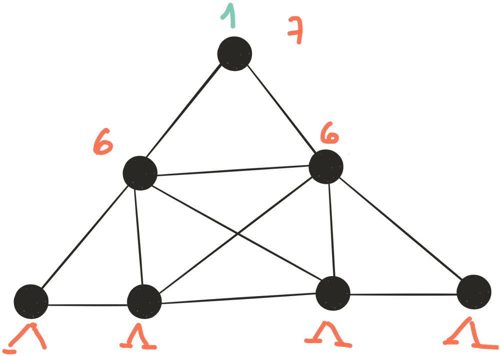

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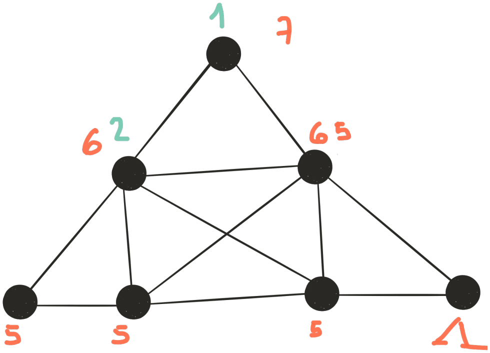

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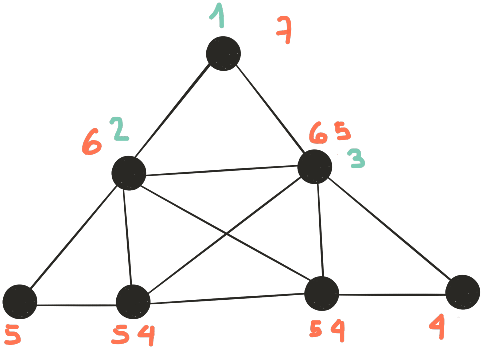

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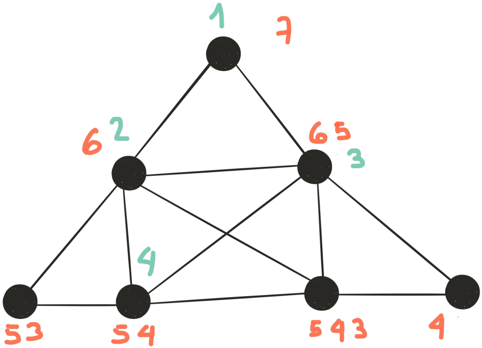

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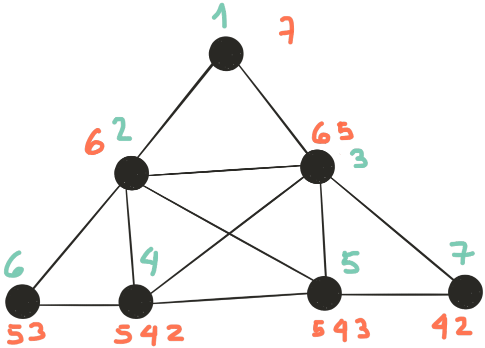

Properties of lexBFS ordering

Dragan, Nicolai & Brandstädt (1997)

An ordering $\prec$ of the vertices of an arbitrary graph G is a lexBFS ordering if and only if for all vertices a, b, c of G such that $ac \in E(C)$ and $bc \notin E(G)$, $c \prec b \prec a$ implies the existence of a vertex d such that d is adjacent to b but not to a and $d \prec c$.

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Rose, Tarjan & Lueker (1976)

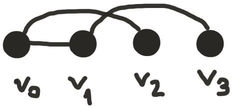
Let σ be a lexBFS ordering of a chordal graph G an let v be an arbitrary vertex of G. Let W be the set of vertices of G that occur before v in σ. Then, v is a simplicial vertex in the induced subgraph $G[\{v\} \cup W]$.

Proof sketch

Arguing towards a contradiction, suppose that $v = v_0$ is not a simplicial vertex of $G[\{v\} \cup W]$. Assume w.o.l.o.g that v is the vertex with the largest index in σ and so $W = V(G) \setminus \{v\}$. .

Proof sketch

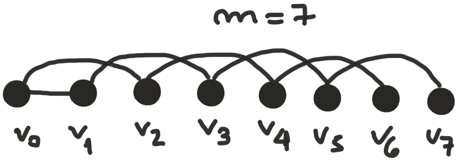
Consequently, there exists two vertices v_1 and v_2 such that $v_1, v_2 \in N_G(v)$ and $v_1v_2 \notin E(G)$. Choose v_2 the smallest (respect to the ordering σ). We know that there exists a vertex v_3, such that v_3 is adjacent to v_2 but not to v_1, choose v_3 as smaller as possible (with respect to σ). Besides, since G is chordal, $v_2v_3 \notin E(G)$



.

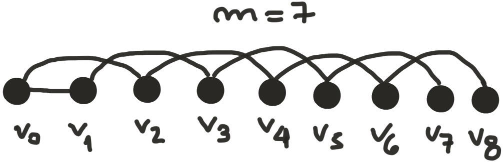
Proof sketch

. Consider the longest decreasing sequence (with respect to σ) $v_0, v_1, v_2, \ldots, v_m$ such that, $v_0 v_i \in E(G)$ for $i = 1, 2$, $v_i v_j \in E(G)$ iff $|i - j| = 2$, and v_j is the smallest vertex (respect to σ) such that v_j is adjacent to v_{j-1} but not to v_{j-2}.



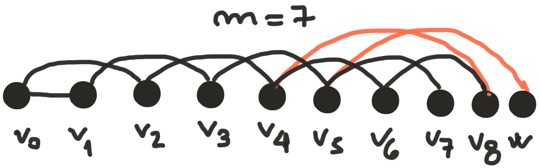
Proof sketch

. We know that there exists a vertex v_{m+1} which is adjacent to v_{m-1} but not to v_{m-2} (chosen as smaller as possible with respect to σ).



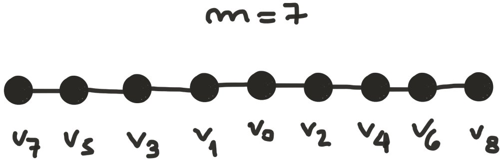
Proof sketch

. Arguing towards a contradiction, suppose that v_{m+1} is adjacent to v_{m-3}. Therefore, there exists a vertex w greater than v_{m+1} (and so greater than v_m) which is adjacent to v_{m-2} but not to v_{m-3}, contradicting that v_m is minimum.



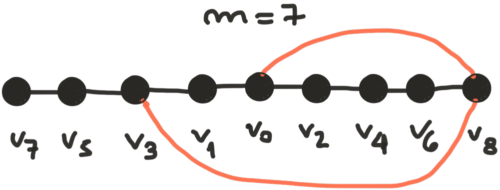
Proof sketch

. Therefore, v_{m+1} is not adjacent to v_{m-3}.



Proof sketch

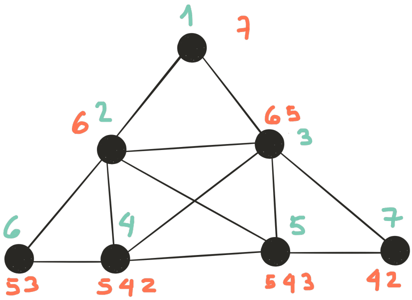
. Finally, since G is chordal, v_{m+1} is not adjacent to v_i for each $i = 0, \ldots, m-4, m$, contradicting the maximality of m.$\square$



Chordal graph linear-time recognition algorithm

- **Input:** An arbitrary graph G.
- **Output:** A statement declaring whether or not G is a chordal graph.

1. Do an arbitrary lexBFS.
2. If the reverse of the lexBFS ordering σ is a perfect vertex elimination ordering, then conclude that G is a chordal graph; else, conclude that G is not a chordal graph.



Gaussian elimination

- Gaussian elimination on an $n \times n$ matrix M consists in choosing a nonzero pivot m_{ij}, then using elementary row and column operations to change m_{ij} into 1, and to change m_{rj} and m_{is} into 0 for all $r \neq i$ and $s \neq j$.

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- An elimination scheme is a sequence of n pivots to reduce a nonsingular matrix M entries to the indentity matrix.
- A perfect elimination scheme has the further property that non zero entry is ever made nonzero along the way.
- Given a symmetric matrix M the graph of M, denoted by $G(M)$, has a vertex set $\{1, 2, \ldots, n\}$ such that i is adjacent to j iff $m_{ij} \neq 0$.

Example

Bad choice:

$$\begin{pmatrix} \mathbf{4} & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathbf{3} & -1 \\ 0 & -1 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 8 \end{pmatrix}$$

Good choice:

$$\begin{pmatrix} \mathbf{4} & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & \mathbf{1} \end{pmatrix} \rightarrow \begin{pmatrix} 4 & 1 & 0 \\ 1 & \mathbf{1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Rose (1970)

A symmetric matrix M with nonzero diagonal entries has a perfect elimination scheme if and only if $G(M)$ is chordal.

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Proof sketch: Pivoting m_{ii} results in removing all the edges incidents to the vertex i and simultaneously adding all edges rs whenever $m_{ri} \neq 0$ and $m_{is} \neq 0$.

$$\begin{pmatrix} & \vdots & & \vdots & \\ \cdots & m_{ii} & \cdots & m_{is} & \cdots \\ & \vdots & & \vdots & \\ \cdots & m_{ri} & \cdots & 0 & \\ & \vdots & & \vdots & \end{pmatrix}$$

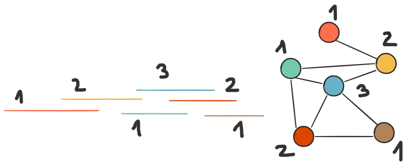
Hence no zero entry is made zero in M precisely when every two neighbors of i are adjacent in $G(M)$; equivalently i is a simplicial vertex of $G(M)$.

Definition

- An interval graph G is an intersection graph of a family $\mathcal{F}$ of closed (or open) intervals in the real line.
- The family $\mathcal{F}$ is called an interval model of G

Application to scheduling

- Consider a collection $C = \{c_i\}$ of courses. Let T_i be the time interval during which course c_i is to take place. Which is the minimum number of classrooms needed to be assigned so that there is no two courses c_i and c_j in the same classroom such that $T_i \cap T_j \neq \emptyset$.



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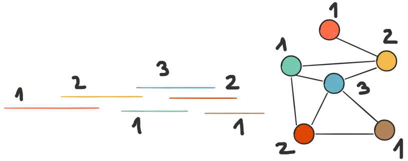
- The problem can be solved by proper coloring the interval graph G, with $V(G) = \{c_i\}_i$ and $E(G) = \{c_i c_j : \ T_i \cap T_j \neq \emptyset\}$.



Coloring an interval graph

- We denote by $\omega(G)$ and $\chi(G)$ to the size of a maximum clique of a graph G and the minimum numbers of colors needed to proper coloring a graph G.

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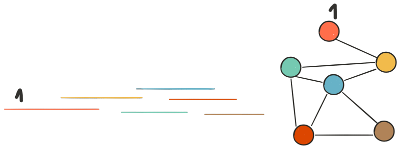

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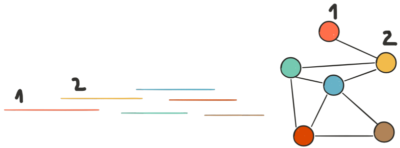

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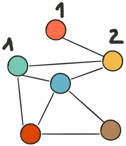

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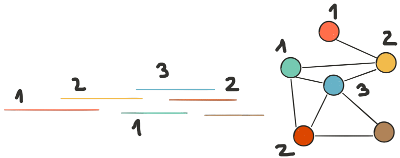

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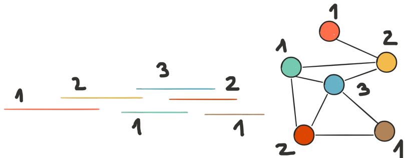

Relationship with chordal graphs

Hajös (1958)

Every interval graph is a chordal graph.

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Proof sketch: Let $\mathcal{F} = \{I_v\}_{v \in V(G)}$ an interval model of G. Arguing towards a contradiction, suppose that G has a cycle $(v_1, v_2, \ldots, v_k)$ with $k \geq 4$. Choose, $p_i \in I_{v_i} \cap I_{v_{i+1}}$ for $i = 1, \ldots, k-1$. Since $I_{v_{i-1}} \cap I_{v_{i+1}} = \emptyset$, $\{p_i\}_{1 \leq i \leq k}$ either is a strictly increasing or a strictly decreasing sequence. Therefore, $I_{v_1} \cap I_{v_k} = \emptyset$ contradicting that $v_1 v_k \in E(G)$.$\square$

Comparability graphs

- A digraph $D = (V, A)$ has a transitive orientation if for each arc $(a, b) \in A$ and $(b, c) \in A$ then $(a, c) \in E$.
- A graph $G = (V, E)$ is called a comparability graph if its edges can be oriented so that the resulting digraph $D = (V, A)$ has a transitive orientation.

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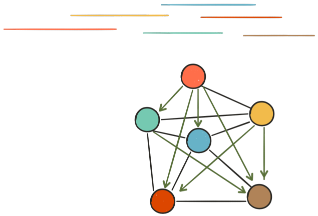
Ghouila-Houri (1962)

If G is an interval graph, then $\overline{G}$ is a comparability graph.

Proof sketch

Let $\{I_v\}_{v \in V(G)}$ be an interval model of G. Define the orientation A of the edges of $\overline{G}$ as follows: $(u, v) \in D$ if and only if I_u is fully to the left of I_v.

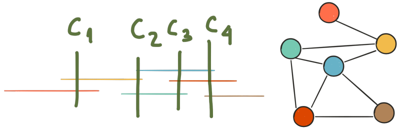


Characterization

Gilmore & Hoffman (1964)

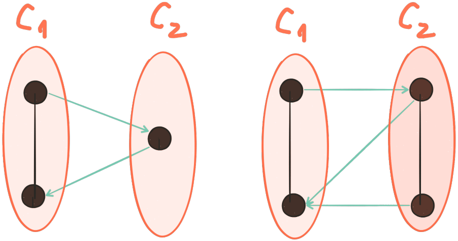
Let G be a graph. The following statements are equivalent:

1. G is an interval graph.
2. G is a chordal graph and its complement $\overline{G}$ is a comparability graph.
3. The maximal cliques of G can be linearly ordered such that, for every vertex v, the cliques containing v occur consecutively.



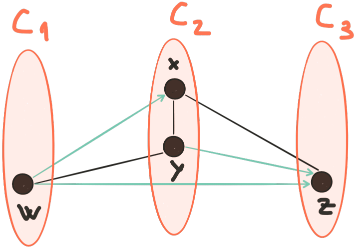
Proof sketch

2.⇒ 3. Let $D = (V, A)$ be a transitive orientation of $\overline{G}$. Given two maximal cliques of G, C_1 and C_2, it can be proved that: i) there exists one arc with one of its endpoints in C_1 and the other one in C_2; and ii) All such arcs in D connecting C_1 and C_2 have the same orientation



Proof sketch

Consider the following relation on the collection $\mathcal{C}$ of maximal cliques of G: $C_1 < C_2$ if there exists an arc in A connecting a vertex in C_1 with a vertex in C_2 oriented from C_1 toward C_2. It can be proved that $(\mathcal{C}, <)$ is a transitive tournament.



Proof sketch

Arguing towards a contradiction, suppose that there exists a vertex v, and $C_i, C_j, C_k \in \mathcal{C}$ such that $C_i < C_j < C_k$ and $v \in C_h$ for $h = i, k$ and $v \notin C_j$. Therefore, there exists a vertex $w \in C_j$ such that $vw \in E(\overline{G})$. In addition, $v \in C_i$ implies $(v, w) \in D$ and $x \in C_k$ implies $(w, v) \in D$, a contradiction.
3.$\Rightarrow$ 1.. For each vertex $v \in V(G)$ we define I_v as the minimal closed interval in $\mathbb{R}$ containing the set of integers $\{i : v \in C_i\}$. It is easy to see that $vw \in E(G)$ if and only if $I_v \cap I_w \neq \emptyset$.$\square$

Clique matrix & consecutive one property

- The clique matrix of a graph G is the maximal cliques versus vertices incident matrix.
- A matrix whose entries are zeros and ones has the consecutive 1's property for columns if its rows can be permuted in such a way that the 1's in each column occur consecutively.

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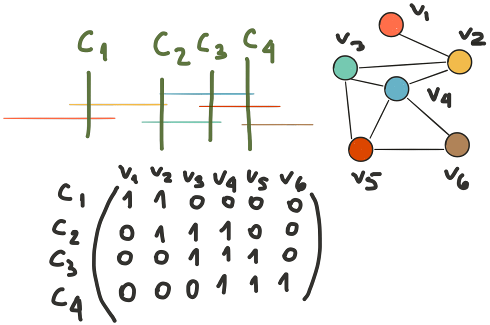
$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Consecutive 1's property for columns

Fulkerson & Gross (1965)

A graph G is an interval graph if and only if its clique matrix M has the consecutive 1's property for columns



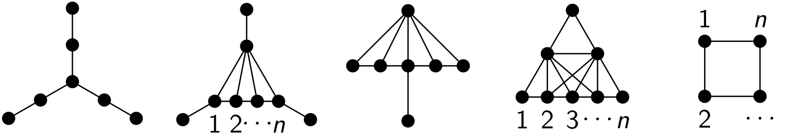
Characterization of Interval Graphs

- Three vertices in a graph G form an asteroidal triple if every two of them are connected by a path avoiding the third one.

Boland and Lekkerkerker (1962)

Let G be a graph, the following statements are equivalent:

1. The graph G is an interval graph.
2. The graph G is chordal and contains no asteroidal triple.
3. The graph G does not contain any of the following graphs as induced subgraphs.



bipartite claw, n-net, $n \geq 2$, umbrella, n-tent, $n \geq 3$, C_n, $n \geq 4$

Another characterization

Olariu (1991)

For a graph G the following two statements are equivalent:

1. G is an interval graph.
2. There exists a linear order $<$ on $V(G)$ such that for every choice of vertices u, v and w with $u < v$ and $v < w$, $uw \in E(G)$ implies $uv \in E(G)$.

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Proof Sketch: 1. $\Rightarrow$ 2. Let $I_u = [u_\ell, u_r]$ and $I_v = [v_\ell, v_r]$. Let G be an interval graph with an interval model $\mathcal{F}$ we define a linear ordering on $V(G)$ in such a way that $u_\ell < v_\ell$, or $u_\ell = v_\ell$ and $u_r \leq v_r$, whenever $u < v$. It can be proved that such a linear ordering on $V(G)$ satisfies the condition 2.

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Proof Sketch: 2. $\Rightarrow$ 1.. We enumerate the vertices of G as $w_1, \ldots, w_n$ in such a way that $w_i < w_j$ whenever $i < j$. First, we will prove that G is chordal. It can be easily proved that $[w_n, w_{n-1} \ldots, w_2, w_1]$ is a perfect vertex elimination scheme.

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Proof Sketch: Finally it remains to prove that $\overline{G}$ is a comparability graph. We define the digraph D in such a way that $V(G) = V(D)$ and $(u, v) \in A(D)$ if and only if $u < v$ and $uv \notin E(G)$. Consequently if $(u, v) \in A(D)$ and $(v, w) \in A(D)$, then $u < w$ and $uw \notin E(G)$ because otherwise $uv \in E(G)$.$\square$

Definitions and characterizations

- A unit interval graph is an interval graph having an interval model $\mathcal{F}$ such that all intervals in $\mathcal{F}$ have the same length. Such an interval model is called a unit interval model of the graph.
- A proper interval graph is an interval graph having an interval model $\mathcal{F}$ with no interval properly contained in another interval. Such an interval model is called a proper interval model of the graph.

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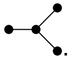
Roberts (1969)

Given an interval graph G the following conditions are equivalent:

1. G is a proper interval graph.
2. G is a unit interval graph
3. G is an interval graph contains no induced claw .

Proof sketch [Bogart & West (1999)]

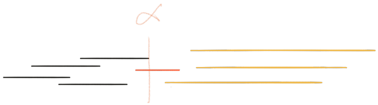
1. and 2. $\Rightarrow$ 3. In an interval representation of the claw the intervals for the three vertices of degree one must be pairwise disjoint, and so the vertex of degree three contains the interval of degree one in the meddle.

Proof sketch [Bogart & West (1999)]

3. $\Rightarrow$ 1. Let G a claw-free interval graph. Since G is claw-free, there is no pair of vertices v and w such that $I_v = [v_1, v_2]$ is properly contained in $I_w = [w_1, w_2]$ such that there is endpoints in $[w_1, v_1]$ and $[v_2, w_2]$ of intervals that do not intersect I_v. Hence we can extend I_v past the end of I_w on one end without changing the graph obtained from the representation. Repeating until no more pairs of intervals are related by inclusion yields a proper interval representation.

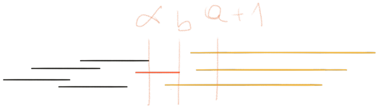
Proof sketch [Bogart & West (1999)]

1. $\Rightarrow$ 2. We process the representation from left to right, adjusting all the intervals to length 1. Let $I_x = [a, b]$ the interval having the leftmost left endpoint. Let $\alpha = a$ unless I_x contains the right endpoint of some other interval, in which case let α be the largest such right endpoint. Hence $\alpha < \min\{a + 1, b\}$.



Proof sketch [Bogart & West (1999)]

Now, adjusting the portion representation of $[a, +\infty)$ by shrinking or expanding $[\alpha, b]$ to $[\alpha, a+1]$ and translating $[b, +\infty)$ to $[a+1, \infty)$.$\square$



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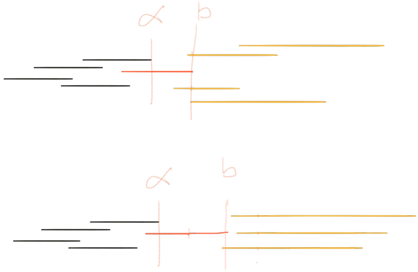
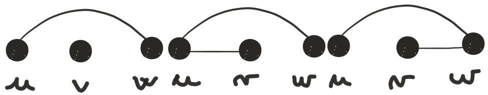


Another characterization

Looges & Olariu (1993)

A graph G is a proper interval graph if and only if there exists a linear ordering on $V(G)$ such that for every choice of $u, v, w \in V(G)$, $u < v < w$ and $uw \in E(G)$ implies that $uv \in E(G)$ and $vw \in E(G)$.

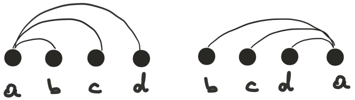


Proof sketch

$\Rightarrow$ Let $I_u = [u_\ell, u_r]$ and $I_v = [v_\ell, v_r]$. Let G be a proper interval graph with an interval model $\mathcal{F}$ we define a linear ordering on $V(G)$ in such a way that $u_\ell < v_\ell$, or $u_\ell = v_\ell$ and $u_r \leq v_r$, whenever $u < v$. It can be proved that such a linear ordering on $V(G)$ satisfies the condition of the necessary condition of the statement.

Proof sketch

$\Leftarrow$ Since the linear ordering on $V(G)$ satisfies the condition of the linear ordering for interval graph of Olariu's characterization, in virtue of Robert's characterization, it suffices to show that G is a claw-free graph. Arguing towards a contradiction, suppose that $\{a, b, c, d\}$ induces a claw in G being a the vertex of degree three. It is easy to see that a cannot precede (follow) b, c, d. We can assume w.o.l.o.g that b precedes a, c, d and d follows a, b, c. Either case leads to a contradiction.$\square$



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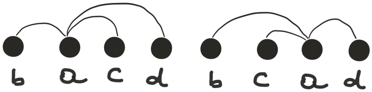


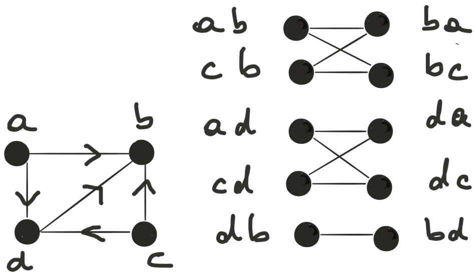
Definitions

- Recall that a comparability graph is a graph having a transitive orientations of its edges.
- A quasi-transitive orientation of a graph G is an orientation $A(G)$ of if edges in such a way that if $(u, v) \in A(G)$ and $(v, w) \in A(G)$, then $u, w \in E(G)$.
- Given a graph G we define G^+ as the graph whose vertex set consists of all ordered pairs (u, v) with $uv \in E(G)$, a vertex (u, v) is adjacent in G^+ to (v, u), to any (w, u) such that $vw \notin E(G)$ and to any (v, w) such that $uw \notin E(G)$

Quasi-transitive orientations

Ghouilà-Houri (1962)

The edges of a graph G can be quasi-transitively oriented if and only if G^+ is a bipartite graph.



Lexicographic-two-coloring of G^+

- A graph G whose vertices are $1, 2, \ldots, n$.
- A two-coloring of G^+

1. while there is an uncolored vertex in G^+ do
2. assign color A to the lexicographic smallest unclored vertex $(u, v) \in G^+$
3. complete to the unique two-coloring the component containing (u, v) assigning the color A to those vertices at even distance from (u, v) and the color B to those vertices at odd distance from (u, v).

Comparability graph

Hell and Huang (1995)

Let D be a quasi-transitive orientation of a quasi-transitive orientable graph G in which $(u, v) \in A(D)$ just if (u, v) obtains the color A in the lexicographic-two-coloring of G^+. Then D is a transitive orientation of G.

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Proof sketch: Arguing towards a contradiction suppose that $\{u, v, w\}$ forms the smallest ordered triple in the lexicographic ordering for which the conditions of transitively fails. Suppose, w.o.l.o.g, that $(u, v), (v, w), (w, u) \in A(D)$ and $v > w$. So (v, w) was not the first vertex colored A in its connected component. Suppose that (v', w') was the first and so $v' < w'$ and $\{u, v', w'\}$ forms an smaller ordered triple than $\{u, v, w\}$, and there is an even length path $(v, w) = (v_0, w_0), (v_1, w_1), \ldots, (v_{2k}, w_{2k}) = (v', w')$ in G^+.

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It can be proved that for each even $(u, v_i), (v_i, w_i), (w_i, u) \in A(D)$ and for each odd $(u, w_i), (w_i, v_i), (v_i, u) \in A(G)$. Therefore, $(u, v_{2k}), (v_{2k}, w_{2k}), (w_{2k}, u) \in A(G)$, contradicting that is $\{u, v, w\}$ is the minimum triple with this property.$\square$

Corollary

Ghouilà-Houri (1962)

A graph G is a comparability graph if and only if G is quasi-transitively orientable

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$O(m\Delta)$ **recognizing algorithm**

1. Construct G^+.
2. While there exists uncolored vertices do color by A the lexiscographically smallest uncored vertex (u, v) use BFS to two-color (if possible) the component of G^+ which contains (u, v)
3. If some component could not be two-colored then report that G is not a comparability graph.
4. Orient the edge uv of G as (u, v) just if (u, v) obtained color A.

Local tournament

- A quasi-transitive orientation of a graph G is an orientation $A(G)$ of if edges in such a way that if $(u, v) \in A(G)$ and $(v, w) \in A(G)$, then $u, w \in E(G)$.
- Given a graph G we define G^* as the graph whose vertex set consists of all ordered pairs (u, v) with $uv \in E(G)$, a vertex (u, v) is adjacent in G^* to (v, u), to any (u, w) such that $vw \notin E(G)$ and to any (w, v) such that $uw \notin E(G)$

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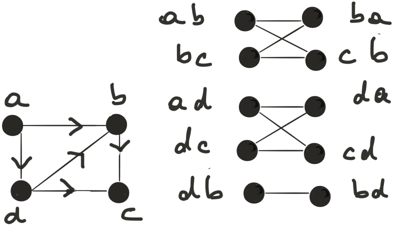
Hell & Hoang (1995)

The graph G has a local tournament orientation if and only if G^* is a bipartite graph.



Proper interval graph

Hell & Hoang (1995)

A graph is a proper interval graph if and only if it is orientable as an acyclic local tournament.

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Proof sketch: $\Rightarrow$ If $\mathcal{F}$ is a proper interval model of G an acyclic local tournament is obtained by oriented the edge uv as (u, v), whenever the interval representing u contains the left endpoint of the interval representing v.

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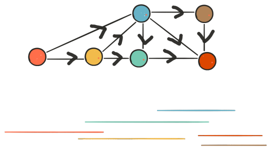
Proof sketch: $\Rightarrow$ If $\mathcal{F}$ is a proper interval model of G an acyclic local tournament is obtained by oriented the edge uv as (u, v), whenever the interval representing u contains the left endpoint of the interval representing v.
$\Leftarrow$ If D is a local tournament orientation of G, the we can enumerate the vertices of D as $v_1, \ldots, v_n$ so tat for each i there exists positive integers k, l such that the vertex v_i has inset $\{v_{i-1}, \ldots, v_{i-k}\}$ and outset $\{v_{i+1}, \ldots, v_{i+l}\}$. Each vertex v_i can be represented as the interval $[j, j + d_j^+ + 1 - \frac{d_j^+}{d_j^+ + 1}]$, where d_j^+ is the outdegree of the vertex j.

Acyclic local tournament

Hoang (1993)

Let G be a proper interval graph and let $1, 2, \ldots, n$ be a vertex perfect elimination scheme. Let D be the orientation of G if in which $(u, v) \in A(D)$ if (u, v) obtains color A in the lexicographic two-coloring of G^*. Then D is an acyclic local tournament of G.



Polynomial-time recognition algorithm

1. Rename the vertices of G so that $1, 2, \ldots, n$ is a perfect vertex elimination scheme. If G does not admit such a PVES, then report G is not a proper interval graph.
2. Construct G^*.
3. While there exist uncolored vertices do color by A the lexicographically smallest uncolored vertex (uv) use BFS to two-color (if possible) the component of G^* which contains (u, v).
4. If some component could not be two-colored then report G is not a proper interval graph.
5. Orient the edge uv as (u, v) if (u, v) obtained color A.
6. If the resulting oriented graph contains a directed cycle then report that G is not a proper interval graph.
7. Construct a proper interval model.

Thank you for your attention!

Bibliography

Kenneth P. Bogart and Douglas B. West.
A short proof that 'proper = unit'.
Discrete Mathematics, 201(1-3):21–23, 1999.

Martin Charles Golumbic.
Algorithmic Graph Theory and Perfect Graphs (Annals of Discrete Mathematics, Vol 57).
North-Holland Publishing Co., Amsterdam, The Netherlands, The Netherlands, 2004.

Pavol Hell and Jing Huang.
Lexicographic orientation and representation algorithms for comparability graphs, proper circular arc graphs, and proper interval graphs.
Journal of Graph Theory, 20(3):361–374, 1995.

Terry A. McKee and F.R. McMorris.
Topics in Intersection Graph Theory.
SIAM Monographs on Discrete Mathematics and Applications, 1999.

Douglas B. West.
Introduction to Graph Theory.
Prentice Hall, 2 edition, 2000.