Agent interactions based models

Nonlocal diffusivities

Mayte Pérez-Llanos,

Instituto de Cálculo

U.Buenos Aires

Part I

Agent interactions based models. Applications to Sociology, Biology, Economy, etc

Introduction to agent based models

- Opinion formation process began to attract attention of the physicist community in the 90's up-to nowadays:

  Amblard, Axelrod, Ben-Naim, Braunstein, Deffuant, Di Mare, Galam, Hegselmann, Krapivsky, Krause, Latora, Nadal, Redner, Slanina, Sznajd, Vázquez, Weisbuch ...

- Use ideas from statistical mechanics considering interactions
  - among ~~particles~~ $\rightsquigarrow$ among individuals
  - exchanging ~~energy~~ $\rightsquigarrow$ exchanging opinion
  - during ~~shocks~~ $\rightsquigarrow$ during encounters.

- C. Castellano, S. Fortunato, V. Loreto, "Statistical physics of social dynamics", Reviews of modern physics, 2009.

  P. Sen, B.K. Chakrabarti, "Sociophysics: an introduction", Oxford Univ. Press, 2014.

Physicists get in

In most of the models the interactions are binary and the opinion is either binary or continuous.

The agents are considered as the nodes of a graph.
When the graph is complete (everybody can interact with everybody) some mean-field equations can be used to get theoretical predictions.

Physicists get in

In most of the models the interactions are binary and the opinion is either binary or continuous.

The agents are considered as the nodes of a graph.
When the graph is complete (everybody can interact with everybody) some mean-field equations can be used to get theoretical predictions.

Some other models add restrictions /modifications to the occurrence of the interactions : bounded confidence, telephoning, relocating, etc.

Physicists get in

In most of the models the interactions are binary and the opinion is either binary or continuous.

The agents are considered as the nodes of a graph.
When the graph is complete (everybody can interact with everybody) some mean-field equations can be used to get theoretical predictions.

Some other models add restrictions /modifications to the occurrence of the interactions : bounded confidence, telephoning, relocating, etc.

Another aspect is the tendency to agreement versus repulsion or versus amplification effects.
Eventually, these effects may produce consensus or on the contrary polarization.

Physicists get in

In most of the models the interactions are binary and the opinion is either binary or continuous.

The agents are considered as the nodes of a graph.
When the graph is complete (everybody can interact with everybody) some mean-field equations can be used to get theoretical predictions.

Some other models add restrictions /modifications to the occurrence of the interactions : bounded confidence, telephoning, relocating, etc.

Another aspect is the tendency to agreement versus repulsion or versus amplification effects.
Eventually, these effects may produce consensus or on the contrary polarization.

Some more realistic (and much more delicate) models consider that the agents are nodes located in complex networks.
Most of the results are based on computer simulations.

Mathematicians get in

This contributed with an important development of the mathematics behind those models, using Boltzmann-like equations and nonlinear PDEs such us Fokker-Planck, porous media, transport equations, which sometimes include nonlocal terms.

- G. Toscani,"Kinetic models of opinion formation", Comm. Math. Sci., 2006.
- L. Pareschi, G. Toscani, "Interacting Multiagent Systems: Kinetic equations and Monte Carlo methods", Oxford University Press, 2013.
- B. During, P. Markowich, J-F. Pietschmann, M-T. Wolfram, "Boltzmann and Fokker-Planck equations modeling opinion formation in the presence of strong leaders", Proc. Royal Society A, 2009.

Within the context of Opinion Formation models we refer to the works due to Aletti, Bellomo, Carrillo, During, Liggett, Naldi, Markowich, Otto, Pareschi, Salvarani, Toscani, Villani.... and a long etc.

A first simple model:

- Tendency to compromise

[Deffuant, Neau, Amblard, Weisbuch, Adv. Complex Systems (2000)]

[Pinasco, Balenzuela, Semeshenko, Chaos Solitons Fractals (2017)]

Opinion dynamics models

- Imagine a group of people is discussing about some question like
  - What do you think of Catalonia independence, for or against ?
  - In a trial, do you find the accused person guilty, yes or no ?
  - Are you against or in favor of abortion?

Each member has an opinion (binary or continuous).

Opinion dynamics models

- Imagine a group of people is discussing about some question like
  - What do you think of Catalonia independence, for or against ?
  - In a trial, do you find the accused person guilty, yes or no ?
  - Are you against or in favor of abortion?

  Each member has an opinion (binary or continuous).

- They influence each other - social pressure [Jenness, A. 1932]

  - Persuasive argument theory:
    [A. Vinokur, E. Burnstein, 1978]
    People pool their arguments together and modify their opinion after examination.
  - Compromise tendency:
    [DeGroot's 1974]
    Individuals tend to go to some kind of mean opinion of the group.

A simple opinion model:

- Assume that each agent is characterized by its opinion $w \in [-1, 1]$.

A simple opinion model:

- Assume that each agent is characterized by its opinion $w \in [-1, 1]$.
- We randomly choose two agents with opinion w and w_* who are to interact.
  As a result, they modify their opinion according to:

$$\begin{aligned} w' &= w + \gamma(w_* - w) \\ w'_* &= w_* + \gamma(w - w_*) \end{aligned}$$

$\gamma > 0$ regulates the strength of the interaction.

A simple opinion model:

- Assume that each agent is characterized by its opinion $w \in [-1, 1]$.
- We randomly choose two agents with opinion w and w_* who are to interact.
  As a result, they modify their opinion according to:

$$\begin{aligned} w' &= w + \gamma(w_* - w) \\ w'_* &= w_* + \gamma(w - w_*) \end{aligned}$$

  $\gamma > 0$ regulates the strength of the interaction.
- This interaction rule models the tendency to compromise.

A simple opinion model:

By f_t we denote the distribution of opinion at time t, a probability measure over $[-1, 1]$.

All of the agents tend to share the same opinion w_∞, namely

$$f_t \to \delta_{w_\infty}$$

A simple opinion model:

By f_t we denote the distribution of opinion at time t, a probability measure over $[-1, 1]$.

All of the agents tend to share the same opinion w_∞, namely

$$f_t \to \delta_{w_\infty}$$

Questions:

1. How to prove this convergence, in which sense does it hold, at which velocity does it occur ?
2. $w_\infty = ???$

Evolution equation

Let us consider a function $\phi : [-1, 1] \to \mathbb{R}$ (an "observable").

How does the mean value $\int_{-1}^{1} \phi(w)\, df_t(w)$ of ϕ evolve in time ?

Evolution equation

Let us consider a function $\phi : [-1, 1] \to \mathbb{R}$ (an "observable").

How does the mean value $\int_{-1}^{1} \phi(w)\, df_t(w)$ of ϕ evolve in time ?

The change of ϕ in an interaction is $\phi(w') - \phi(w)$.

We take the mean over all possible changes to get

$$\frac{d}{dt} \int_{-1}^{1} \phi(w)\, df_t(w) = \int_{-1}^{1} \int_{-1}^{1} \Big(\phi(w') - \phi(w)\Big)\, df_t(w)\, df_t(w_*) \quad \forall\, \phi \in C^\infty.$$

It is a Boltzmann-like equation.

Evolution of the mean

How does the mean opinion

$$m(t) = \int_{-1}^{1} w \, df_t(w)$$

evolve?

Evolution of the mean

How does the mean opinion

$$m(t) = \int_{-1}^{1} w \, df_t(w)$$

evolve?

We take $\phi(w) = w$ and recall that $w' = w + \gamma(w_* - w)$:

$$\begin{aligned} m'(t) &= \gamma \int_{-1}^{1} \int_{-1}^{1} (w_* - w) \, df_t(w) \, df_t(w_*) \\ &= \gamma \Big\{ \int_{-1}^{1} w_* df_t(w_*) - \int_{-1}^{1} w df_t(w) \Big\} = 0. \end{aligned}$$

Thus the mean opinion is conserved: $m(t) = m(0)$.

Evolution of the variance

- How does the variance evolve ?:

$$\sigma^2(t) = \int_{-1}^{1} (w - m(t))^2 \, df_t(w) = \int_{-1}^{1} w^2 \, df_t(w) - m(0)^2,$$

Evolution of the variance

- How does the variance evolve ?:

$$\sigma^2(t) = \int_{-1}^{1} (w - m(t))^2 \, df_t(w) = \int_{-1}^{1} w^2 \, df_t(w) - m(0)^2,$$

- Choosing $\phi(w) = w^2$, and recalling that $w' = w + \gamma(w_* - w)$

$$\begin{aligned} (\sigma^2)'(t) &= \frac{d}{dt} \int_{-1}^{1} w^2 \, df_t = \int_{-1}^{1} \int_{-1}^{1} \left[(w')^2 - w^2 \right] df_t(w) df_t(w_*) \\ &= (2\gamma^2 - 2\gamma) \Big\{ \int_{-1}^{1} w^2 \, df_t(w) - m(0)^2 \Big\} = -2\gamma(1-\gamma)\sigma^2(t), \end{aligned}$$

hence $\sigma^2(t) = \sigma^2(0) e^{-2\gamma(1-\gamma)t} \overset{t \to +\infty}{\longrightarrow} 0.$

Evolution of the variance

- How does the variance evolve ?:

$$\sigma^2(t) = \int_{-1}^{1} (w - m(t))^2 \, df_t(w) = \int_{-1}^{1} w^2 \, df_t(w) - m(0)^2,$$

On the other hand,

$$\sigma^2(t) = \int_{-1}^{1} (w - m(0))^2 \, df_t(w) = \iint (w-u)^2 d(f_t, \delta_{m_0})(w,u)$$
$$= W^2(f_t, \delta_{m(0)}) \to 0 \text{ exponentially fast.}$$

- Choosing $\phi(w) = w^2$, and recalling that $w' = w + \gamma(w_* - w)$

$$\begin{aligned}(\sigma^2)'(t) &= \frac{d}{dt}\int_{-1}^{1} w^2 \, df_t = \int_{-1}^{1}\int_{-1}^{1} \left[(w')^2 - w^2\right] df_t(w) df_t(w_*) \\ &= (2\gamma^2 - 2\gamma)\Big\{ \int_{-1}^{1} w^2 \, df_t(w) - m(0)^2 \Big\} = -2\gamma(1-\gamma)\sigma^2(t),\end{aligned}$$

hence $\sigma^2(t) = \sigma^2(0) e^{-2\gamma(1-\gamma)t} \overset{t\to+\infty}{\longrightarrow} 0.$

Second social model:

- Tendency to compromise
- Persuasion
- Stubborness

[P-L, Pinasco, Saintier, Silva, SIAM Math. Analysis (2018)]

Heterogeneous persuasion and zealotry

Let us assume that each agent is characterized by

- her opinion $w \in [-1, 1]$,

Heterogeneous persuasion and zealotry

Let us assume that each agent is characterized by

- her opinion $w \in [-1, 1]$,
- her persuasion capacity $p \in [0, 1]$:
  $p = 0$: incapable of convincing to anybody
  $p = 1$: the best speaker ever

Heterogeneous persuasion and zealotry

Let us assume that each agent is characterized by

- her opinion $w \in [-1, 1]$,
- her persuasion capacity $p \in [0, 1]$:
  $p = 0$: incapable of convincing to anybody
  $p = 1$: the best speaker ever
- her willingness to change opinion $q \in [0, 1]$:
  $q = 0$: stubborn agent (she will never change her opinion whatever she is being told)
  $q = 1$: changes his opinion very easily

Heterogeneous persuasion and zealotry

Let us assume that each agent is characterized by

- her opinion $w \in [-1, 1]$,
- her persuasion capacity $p \in [0, 1]$:
  $p = 0$: incapable of convincing to anybody
  $p = 1$: the best speaker ever
- her willingness to change opinion $q \in [0, 1]$:
  $q = 0$: stubborn agent (she will never change her opinion whatever she is being told)
  $q = 1$: changes his opinion very easily

2 agents (w, p, q), (w_*, p_*, q_*) modify their opinion according to the rule

$$\begin{aligned} w' &= w + \gamma p^* q (w_* - w) \\ w'_* &= w + \gamma p q^* (w - w_*), \end{aligned}$$

$\gamma > 0$ regulates the strength of the interaction.
The parameters (p, q) do not change for each of the agents.

Finding the PDE: A nonlocal transport equation

Asymptotically (via *Grazing limit*) f a weak solution to the following non-local transport equation for probability measure-valued functions,

$$\partial_t f_t + \partial_w \Big((m_t - w) q \langle p \rangle f_t \Big) = 0,$$

where $\langle p \rangle$ is the mean value of p (constant in time) and

$$m_t = \int_K \frac{p}{\langle p \rangle} w \, df_t(w, p, q),$$

the mean opinion weighted by the normalized power of conviction.

Finding the PDE: A nonlocal transport equation

Asymptotically (via *Grazing limit*) f a weak solution to the following non-local transport equation for probability measure-valued functions,

$$\partial_t f_t + \partial_w \Big((m_t - w) q \langle p \rangle f_t \Big) = 0,$$

where $\langle p \rangle$ is the mean value of p (constant in time) and

$$m_t = \int_K \frac{p}{\langle p \rangle} w \, df_t(w, p, q),$$

the mean opinion weighted by the normalized power of conviction.
IF we knew that there exists

$$m_\infty := \lim_{t \to +\infty} m_t,$$

then, in view of the equation, it seems reasonable to conjecture

$$f_{t|q>0}(w) dw \to \delta_{m_\infty}(w).$$

Consensus result

Theorem:

If initially

$$f_{|t=0} = \alpha f_0^0 + (1-\alpha) f_{t=0}^1$$

where $\alpha \in (0,1)$, f_0^0 has support on $\{q = 0\}$ while $f_{t=0}^1$ has support on $\{q \geqslant \varepsilon\}$ ($+$ regularity assumptions), then

$$W_1\left(f_t^1, \delta_{m_\infty}(w) \otimes f_{|t=0}^1(p,q)dpdq\right) \leqslant 4e^{-\alpha\langle p\rangle \varepsilon t}$$

where W_1 denotes the Wasserstein distance and

$$m_\infty := \lim_{t\to+\infty} m_t = \int \frac{p}{\langle p\rangle_{|q=0}} w \, df_0^0(w,p)$$

is the mean opinion of the stubborn agents weighted by their normalized power of conviction.

One more social model:

- Tendency to compromise
- Amplification

Work in progress with

J.P. Pinasco, N. Saintier, R. Tyson

Amplification

Our position with respect an issue has to aspects

- Our attitude: against/ in favor (negative/positive)

Amplification

Our position with respect an issue has to aspects

- Our attitude: against/ in favor (negative/positive)
- Our intensity : $|w|$

Amplification

Our position with respect an issue has to aspects

- Our attitude: against/ in favor (negative/positive)
- Our intensity : $|w|$

Two agents with the same attitude may reinforce their opinions intensity:

$$w' = w + (1 - p_A)h\text{sign}(w^* - w) + p_A h\delta(w, w^*)$$

$$w'_* = w_* + (1 - p_A)h\text{sign}(w - w^*) + p_A h\delta(w, w^*),$$

where
$$\delta(w, w^*) = \begin{cases} 1 & \text{if } w, w^* > 0 \\ -1 & \text{if } w, w^* < 0, \\ 0 & \text{if } w \cdot w^* < 0 \end{cases}$$

B. O. Baumgaertner, P. A. Fetros, S. M. Krone, & R. C. Tyson,"Spatial opinion dynamics and the effects of two types of mixing", Phys. Rev. E, 2018,.

Amplification

Our position with respect an issue has to aspects

- Our attitude: against/ in favor (negative/positive)
- Our intensity : $|w|$

Two agents with the same attitude may reinforce their opinions intensity:

$$w' = w + (1 - p_A)h\text{sign}(w^* - w) + p_A h\delta(w, w^*)$$

$$w'_* = w_* + (1 - p_A)h\text{sign}(w - w^*) + p_A h\delta(w, w^*),$$

where
$$\delta(w, w^*) = \begin{cases} 1 & \text{if } w, w^* > 0 \\ -1 & \text{if } w, w^* < 0, \\ 0 & \text{if } w \cdot w^* < 0 \end{cases}$$

B. O. Baumgaertner, P. A. Fetros, S. M. Krone, & R. C. Tyson,"Spatial opinion dynamics and the effects of two types of mixing", Phys. Rev. E, 2018,.

To pass to a continuous model, which is the attitude at zero???

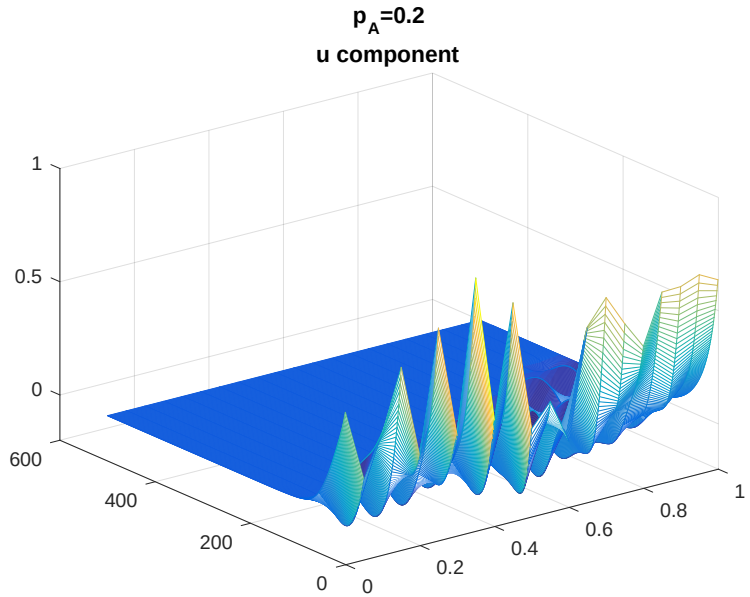
$$s^-(0, t) \leftrightarrows s^+(0, t).$$

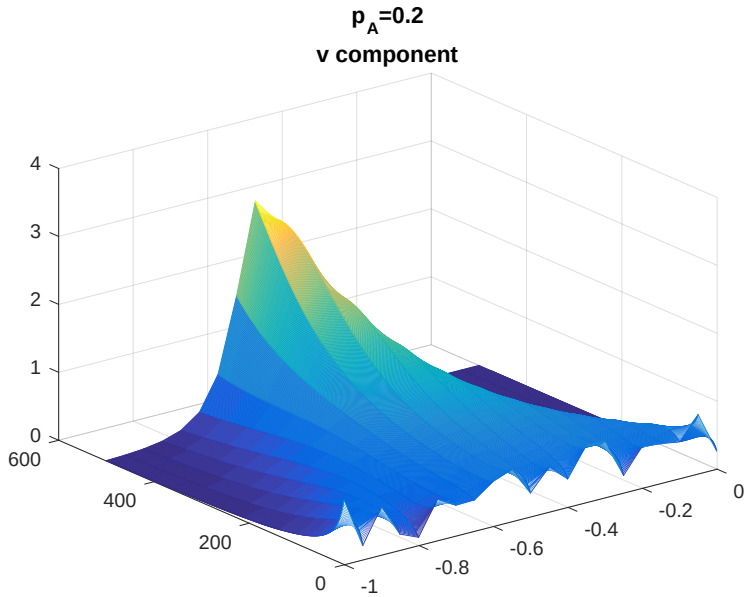
Asymptotic Behaviour: Thereshold $p_A = 1/2$

If we could consider $(s^-, s^+) \sim f_t$ as a density function defined on $[-1, 1]$, the equation *'could be written'*

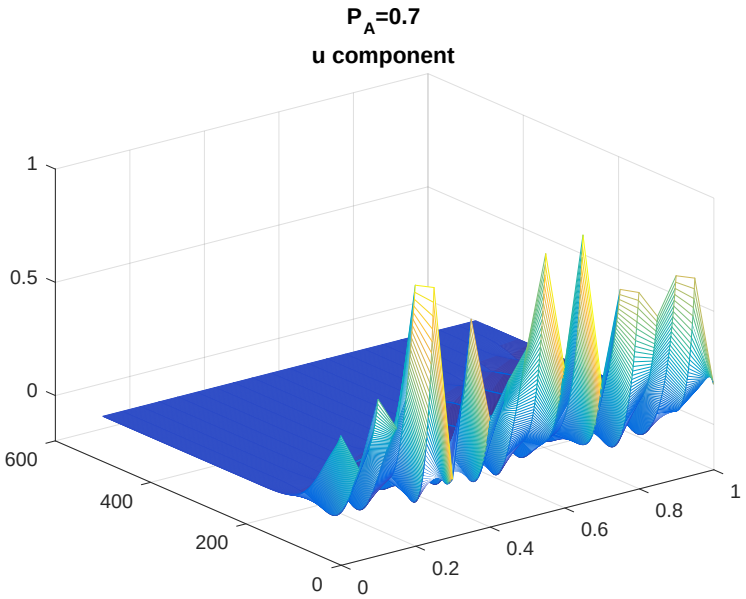
$$\partial_t f_t = (1 - p_A)\partial_x\Big(f_t[2F_t - 1]\Big) + p_A\Big[\int_{-1}^{0} f_t(y)dy - \int_{0}^{1} f_t(y)dy\Big].$$

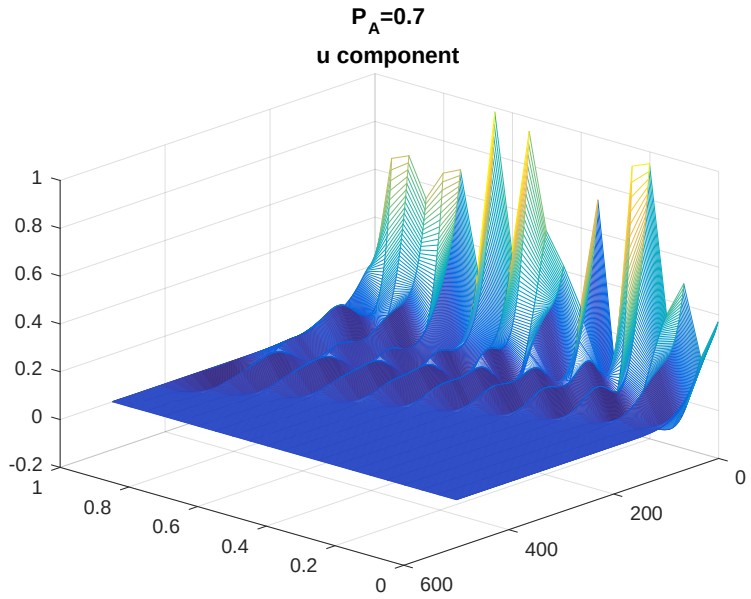
The second term moves the consensus from the initial mean opinion to a more extreme position.

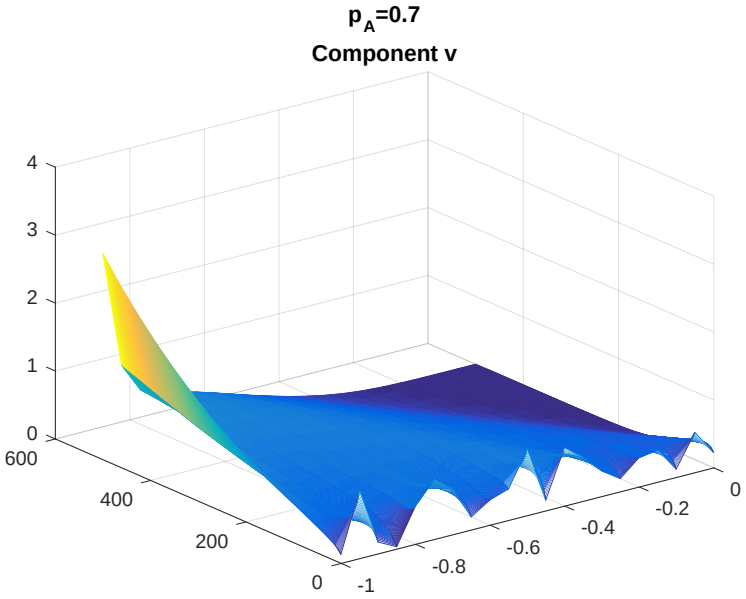


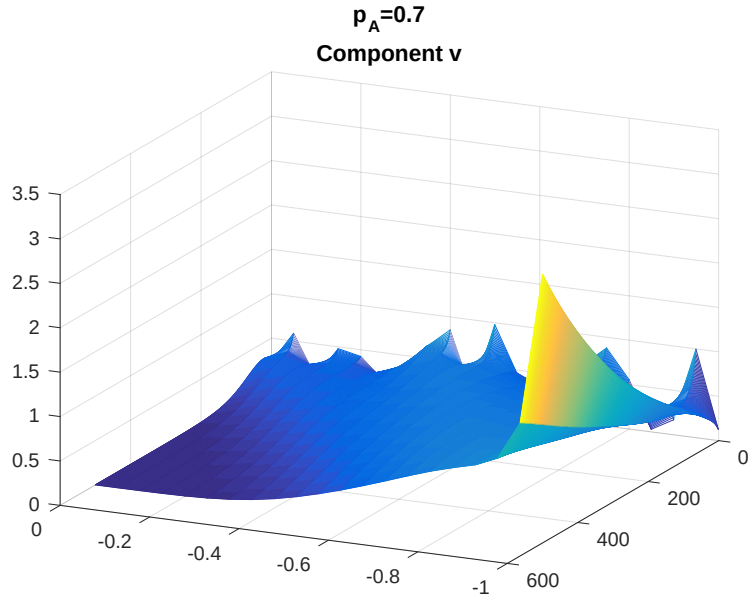


Asymptotic Behaviour: Thereshold $p_A = 1/2$









A Biology model:

- Cell attraction with certain fitness
- Random motion

Work in progress with

P. Balenzuela, M. Cuenca, L. Pedraza, J.P. Pinasco, N. Saintier

Certain tumoral cell motion rules

Experiments show that certain tumoral cells exert certain attraction to other cells.
This attraction varies depending on the position of the attracting cell.
On the other hand, a probability of random motion is observed at every cell.

Certain tumoral cell motion rules

Experiments show that certain tumoral cells exert certain attraction to other cells.
This attraction varies depending on the position of the attracting cell.
On the other hand, a probability of random motion is observed at every cell.
Denoting as x and x^* the positions of the interacting cells, they evolve according the interaction rules

$$\begin{aligned} x' &= x + \gamma\lambda(x^*)(x_* - x) + \sigma^2 Y D(|x|), \\ x'_* &= x_* + \gamma\lambda(x)(x - x_*) + \sigma^2 Y^* D(|x_*|), \end{aligned}$$

where Y, Y_* iid random variable with mean 0 and variance 1.

The function $\lambda(x^*)$ represents the fitness to position x^*.

Project entitled:

Social Dynamics and PDEs

- J.A. Carrillo de la Plata, Imperial College
- M. Ottobre, Heriot Watt University
- M. Pérez-Llanos, University of Buenos Aires
- J.P. Pinasco, University of Buenos Aires
- N. Saintier, University of Buenos Aires
- B. Zegarlinski, Imperial College

Part II

A nonlocal diffusivity

An example of nonlocal diffusion:

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and u_0 an initial population of fleas.

An example of nonlocal diffusion:

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and u_0 an initial population of fleas.

The fleas jump from position x to position y with probability $J(y-x)$, being $J : \mathbb{R} \to \mathbb{R}$ a probability distribution

- $J \geqslant 0$,
- $\int_{\mathbb{R}^N} J(z)dz = 1$,
- $J(z) = J(-z)$.

An example of nonlocal diffusion:

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and u_0 an initial population of fleas.

The fleas jump from position x to position y with probability $J(y-x)$, being $J : \mathbb{R} \to \mathbb{R}$ a probability distribution

- $J \geqslant 0$,
- $\int_{\mathbb{R}^N} J(z)dz = 1$,
- $J(z) = J(-z)$.

The amount of fleas at the point $x \in \Omega$ at time t consists of

- the ones arriving to x from any y:

$$+ \int J(x-y)u(y,t)dy \ ,$$

- minus the ones leaving x to go elsewhere y:

$$- \int J(y-x)u(x,t)dy = - \int J(x-y)u(x,t)dy.$$

An example of nonlocal diffusion:

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and u_0 an initial population of fleas.

The fleas jump from position x to position y with probability $J(y-x)$, being $J : \mathbb{R} \to \mathbb{R}$ a probability distribution

- $J \geqslant 0$,
- $\int_{\mathbb{R}^N} J(z)dz = 1$,
- $J(z) = J(-z)$.

The amount of fleas at the point $x \in \Omega$ at time t consists of

- the ones arriving to x from any y:

$$+ \int J(x-y)u(y,t)dy \ ,$$

- minus the ones leaving x to go elsewhere y:

$$- \int J(y-x)u(x,t)dy = - \int J(x-y)u(x,t)dy.$$

Namely: $$u_t(x,t) = \int J(x-y)(u(y,t)-u(x,t))dy$$ nonlocal!!.

Boundary conditions

- If we do not allow the individuals to enter neither to leave the domain Ω:

$$\int_{\Omega} J(x-y)(u(y)-u(x))dy \quad x \in \Omega,$$

equivalent to an homogeneous Neumann boundary condition on $\partial\Omega$.

Boundary conditions

- If we do not allow the individuals to enter neither to leave the domain Ω:

$$\int_{\Omega} J(x-y)(u(y)-u(x))dy \quad x \in \Omega,$$

equivalent to an homogeneous Neumann boundary condition on $\partial\Omega$.

- To consider a Dirichlet boundary condition g,
  - if $\text{supp}(J) = \mathbb{R}^N$, we need to prescribe g in $\mathbb{R}^N \setminus \Omega$;
  - if $\text{supp}(J) = B_R(0)$, the datum g must be specified in $\Omega_J \setminus \Omega$ being $\Omega_J := \cup_{x\in\overline{\Omega}} B_R(x)$.

Relation with the Laplacian:

In the case of admitting jumps of length ϵ with $\epsilon \to 0$ with an appropriate velocity related to the parabolic scale ($t \sim x^2$), we recover the Laplacian :

$$\Delta u = \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} u.$$

Precisely, the kernel needs to be symmetric and rescaled as follows

$$J_\varepsilon(z) = \frac{1}{\varepsilon^{N+2}} C^{-1} J(z/\varepsilon), \quad C = \frac{1}{2} \int_B J(\theta)\theta_N^2 dz.$$

[Cortázar, Elgueta, Rossi, Wolanski, Arch. Ration. Mech. Anal. (2008)]

$$`\lim_{\varepsilon \to 0} \int_B J_\varepsilon(z)(u(x+\varepsilon z) - u(x))dz = \Delta u(x),'$$

in the sense that

$$\lim_{\varepsilon \to 0} \|u_\varepsilon - u\|_{L^\infty(\Omega)} \to 0.$$

Differences and analogies with the Laplacian

Analogies

- Infinite propagation speed
- Maximum principles

Differences

- Non regularizing effect
- Lack of compactness
- Non existence of eigenvalues ($p \neq 2$)

Non local dispersal

- Population dynamics
- Nerve propagation
- Epidemiology
- Combustion propagation
- Bacterial growth

Some authors

- P. Fife
- J. Coville
- J.A. Carrillo
- S. Martínez
- J. Dávila

Our contributions

Combustion Theory:

- P-L, Rossi, Nonlinear Analysis (2009)
- Ferreira, P-L, Nonlinear Analysis (2012)

Population Dynamics (large solutions):

- Ferreira, P-L, Adv. Nonlinear Studies (2017)

Sandpiles growth ($p \to \infty$):

- P-L, Rossi SIAM J. Numer. Anal. (2011)

Median property ($p \to 1$):

- Mazón, P-L, Rossi, Toledo, Pub. Mat. (2016)

Mil gracias por escuchar !!

Ojalá encontremos

alguna dirección de trabajo !!