Propiedad de Helly, transversal, cover, packing y partition

Problemas de Grafos y Tratabilidad Computacional

Definiciones

Dada una familia de subconjuntos $\mathcal{F} = \{S_1, \cdots, S_n\}$, llamamos $U = \bigcup_{S_i \in \mathcal{F}} S_i$ su universo asociado.

- $\mathcal{F}$ es intersecante si $\forall S_i, S_j \in \mathcal{F}$, $S_i \cap S_j \neq \emptyset$.
- $\mathcal{F}$ verifica «propiedad de Helly» si para toda subfamilia intersecante $\mathcal{F}' \subseteq \mathcal{F}$ cumple $\bigcap_{S_i \in \mathcal{F}'} S_i \neq \emptyset$.
- Un subconjunto $T \subseteq U$ es transversal de $\mathcal{F}$ si $\forall S_i \in \mathcal{F}$, $T \cap S_i \neq \emptyset$.
- Una subfamilia $\mathcal{F}' \subseteq \mathcal{F}$ es set-cover si $U = \bigcup_{S_i \in \mathcal{F}'} S_i$.
- Una subfamilia $\mathcal{F}' \subseteq \mathcal{F}$ es set-packing si $\forall S_i, S_j \in \mathcal{F}'$, $S_i \cap S_j \neq \emptyset \Leftrightarrow i = j$.
- Una subfamilia $\mathcal{F}' \subseteq \mathcal{F}$ es set-partition si es al mismo tiempo set-cover y set-packing.

Algoritmo de Berge

Dada una familia de subconjuntos $\mathcal{F} = \{S_1, \cdots, S_n\}$ y $U = \bigcup_{S_i \in \mathcal{F}} S_i$ su universo asociado.

1. Para cada subconjunto de 3 elementos $T \subseteq U$ hacer
2. Generar $\mathcal{F}_T = \{S_i \in \mathcal{F} \ / \ |S_i \cap T| \geq 2\}$
3. Si $\mathcal{F}_T \neq \emptyset$ y $\bigcap_{S_i \in \mathcal{F}_T} S_i = \emptyset$ entonces devolver Falso
4. Devolver Verdadero

Preguntas:

- ¿Para qué sirve el algoritmo?
- ¿Cuál es la complejidad?
- ¿Por qué es correcto?

Más definiciones

Sea $\mathcal{F} = \{N[v] \ / \ v \in V\}$ la familia de vecindad cerrada de un grafo $G = (V, E)$.

- G es closed neighborhood-Helly si $\mathcal{F}$ cumple propiedad de Helly.
- G es closed neighborhood-Helly hereditario si todos sus subgrafos inducidos son neighborhood-Helly.
- El problema de set-cover mínimo en este caso se llama conjunto dominante mínimo. (dominación)
- El problema de transversal mínimo también mapea al problema de dominación de vértices.
- El problema de set-packing máximo en este caso es el problema de conjunto independiente máximo para G^2.
- El problema de set-partition en este caso se llama dominación eficiente.

Podemos considerar también familias de otros tipos de estructuras: vecindad abierta, clique, etc.

Teorema

Los siguientes problemas son equivalentes en términos de complejidad.

1. El problema de set-cover mínimo para una familia $\mathcal{F}$ de subconjuntos.
2. El problema dominación de vértices para un grafo split G.
3. El problema clique-transversal para un grafo split G.

Otras definiciones

Dado un grafo $G = (V, E)$

- Denotamos $N[v, w]$ la intersección de las vecindades cerradas de v y w ($N[v, w] = N[v] \cap N[w]$).
- Dados $v, w \in V$ y $v \neq w$, definimos su conjunto universal como $U[v, w] = \{u \in V \ / \ N[v, w] \subseteq N[u]\}$. El conjunto universal de vértices $v, w, z \in V$ es $U[v, w, z] = U[v, w] \cap U[w, z] \cap U[z, v]$.
- Definimos la extensión de vértices $v, w, z \in V$ es $E(v, w, z) = N[v, w] \cup N[w, z] \cup N[z, v]$ si $N[v, w], N[w, z], N[z, v] \neq \emptyset$ y $E(v, w, z) = \emptyset$ caso contrario.
- Sean $C_1, \cdots, C_k$ y $v_1, \cdots, v_n$ los cliques y los vértices de G, respectivamente. Llamamos $M \in \{0, 1\}^{k \times n}$ la matriz clique de G donde $m_{ij} = 1$ si $v_j \in C_i$. Llamamos A^* la matriz de adyacencia aumentada de G que es la matriz de adyacencia A de G agregando 1's en la diagonal principal.

Más definiciones

- Una matriz de permutación se obtiene de realizar permutaciones de filas de la matriz identidad.
- Una matriz de co-permutación es el resultado de invertir los ceros y unos de una matriz de permutación.

Teorema

Dado un grafo $G = (V, E)$, las siguientes afirmaciones son equivalentes.

1. G es clique-Helly.
2. Para cada extensión $E(v, w, z)$ donde v, w, z inducen un triángulo de G, existe $u \in V$ tal que $E(v, w, z) \subseteq N[u]$.
3. Para cada extensión $E(v, w, z) \neq \emptyset$ donde v, w, z inducen un triángulo de G, entonces $U[v, w, z] \neq \emptyset$.

Teorema

Dado un grafo $G = (V, E)$, las siguientes afirmaciones son equivalentes.

1. G es clique-Helly hereditario.
2. G no tiene como subgrafos inducidos a los grafos oculares (3-sun,$\overline{P_2 + P_4}$,$\overline{E}$,$\overline{3K_2}$).
3. Para cada extensión $E(v, w, z)$ donde v, w, z inducen un triángulo de G, existe $u \in \{v, w, z\}$ tal que $E(v, w, z) \subseteq N[u]$.
4. Para cada extensión $E(v, w, z) \neq \emptyset$ donde v, w, z inducen un triángulo de G, entonces $U[v, w, z] \cap \{v, w, z\} \neq \emptyset$.
5. La matriz clique M de G no contiene ninguna submatriz de co-permutación de 3×3.

Teorema

Dado un grafo $G = (V, E)$, las siguientes afirmaciones son equivalentes.

1. G es closed neighborhood-Helly.
2. Para cada extensión $E(v, w, z)$ de G, existe $u \in V$ tal que $E(v, w, z) \subseteq N[u]$.
3. Para cada extensión $E(v, w, z) \neq \emptyset$ de G, entonces $U[v, w, z] \neq \emptyset$.

Teorema

Dado un grafo $G = (V, E)$, las siguientes afirmaciones son equivalentes.

1. G es closed neighborhood-Helly hereditario.
2. G no tiene como subgrafos inducidos a C_4, C_5, C_6 ni 3-sun.
3. Para cada extensión $E(v, w, z)$ de G, existe $u \in \{v, w, z\}$ tal que $E(v, w, z) \subseteq N[u]$.
4. Para cada extensión $E(v, w, z) \neq \emptyset$ de G, entonces $U[v, w, z] \cap \{v, w, z\} \neq \emptyset$.
5. La matriz adyacencia aumentada A^* de G no contiene ninguna submatriz de co-permutación de 3×3.

Teorema

Dado un grafo $G = (V, E)$, las siguientes afirmaciones son equivalentes.

1. G es open neighborhood-Helly.
2. G es K_3-free y para cada extensión $E(v, w, z)$ donde v, w, z inducen un co-triángulo de G, existe $u \in V$ tal que $E(v, w, z) \subseteq N[u]$.
3. G es K_3-free y para cada extensión $E(v, w, z) \neq \emptyset$ donde v, w, z inducen un co-triángulo de G, entonces $U[v, w, z] \neq \emptyset$.

Teorema

Dado un grafo $G = (V, E)$, las siguientes afirmaciones son equivalentes.

1. G es open neighborhood-Helly hereditario.
2. G no tiene como subgrafos inducidos a C_6 ni triángulo.
3. G es K_3-free y para cada extensión $E(v, w, z)$ inducen un co-triángulo de G, existe $u \in \{v, w, z\}$ tal que $E(v, w, z) \subseteq N[u]$.
4. G es K_3-free y para cada extensión $E(v, w, z) \neq \emptyset$ inducen un co-triángulo de G, entonces $U[v, w, z] \cap \{v, w, z\} \neq \emptyset$.
5. La matriz adyacencia A de G no contiene ninguna submatriz de co-permutación de 3×3.

Reconocimiento-Status

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open neighborhood-Helly	$O(\Delta n^3)$ (this work)	$O(mn^3)$ [2]
hereditary open neighborhood-Helly	$O(mn^2)$ (this work)	$O(m^2n^2)$ [4]

CHARACTERIZATIONS AND RECOGNITIONS FOR NEIGHBORHOOD HELLY GRAPHS

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1 INTRODUCTION

Denote by G a finite simple graph, with vertex set $V(G)$ and edge set $E(G)$. We use n and m to denote $|V(G)|$ and $|E(G)|$. A *complete set* is a subset $V' \subseteq V(G)$ formed by pairwise adjacent vertices and a *clique* is a maximal complete set. A *triangle* is a complete set of size 3 and a subset of vertices is a *co-triangle* when it is a triangle in $\overline{G}$, the complement of G. Denote by $N(v_i) = \{v_j \in V(G) | (v_i, v_j) \in E(G)\}$, and $N[v_i] = N(v_i) \cup \{v_i\}$, the *open* and *closed neighborhoods* of G, respectively. The degree of a vertex v_i, $d(v_i)$, is $|N(v_i)|$ and the maximum degree of G is denoted by Δ. For $V' \subseteq V(G)$, $G[V']$ is the subgraph of G induced by V'.
Let $\mathcal{F}$ be a family of subsets of some set. Say that $\mathcal{F}$ is *intersecting* when the subsets of $\mathcal{F}$ pairwise intersect. On the other hand, when every intersecting subfamily of $\mathcal{F}$ has a common element then $\mathcal{F}$ is a *Helly* family. The best general algorithm for recognizing Helly families is due by Berge [2].
A graph G is *clique-Helly* when its family of cliques is Helly. Similarly, G is *open neighborhood-Helly (closed neighborhood-Helly)* when its family of open neighborhoods (closed neighborhoods) is Helly. Finally, G is *hereditary clique-Helly (hereditary open neighborhood-Helly, hereditary closed neighborhood-Helly)* when every of its induced subgraphs is clique-Helly (open neighborhood-Helly, closed neighborhood-Helly).
Different characterizations were given for these six graph classes and most of them lead to polynomial-time recognition algorithms: clique-Helly graphs [3, 8]; hereditary clique-Helly graphs [7, 9]; closed neighborhood-Helly graphs [3, 5]; open neighborhood-Helly graphs [5]; hereditary closed neighborhood-Helly graphs [4]; hereditary open neighborhood-Helly graphs [4].
The extension type characterizations for clique-Helly graphs of [1, 8] and for hereditary clique-Helly graphs of [7] have been recently reformulated in [6] (Theorem 2.1 and Theorem 2.4, respectively) leading to more efficient recognition algorithms.
In this work, we describe different characterizations, based on the concept of extensions, for the classes of neighborhoood-Helly graphs, open and closed. In addition, we also describe characterizations for their corresponding hereditary classes. These results are describes in Section 2.
These characterizations lead to new recognition algorithms for neighborhood-Helly graph classes. Finally, in Section 3, we describe some matrix characterizations for hereditary Helly classes.

2 EXTENSION TYPE CHARACTERIZATIONS

First, we need some additional definitions.
Denote by $N[v, w]$ the intersection of $N[v]$ and $N[w]$, i.e. $N[v, w] = N[v] \cap N[w]$. On the other hand, we define the universal set of v, w as: $U[v, w]= \{u \in V(G)/N[v, w] \subseteq N[u]\}$. The universal set of u, v, w is defined as: $U[u, v, w]= U[u, v] \cap U[v, w] \cap U[u, w]$.
Define the *extension* of vertices u, v, w, as $E(u, v, w)= N[u, v] \cup N[v, w] \cup N[u, w]$, whenever $N[u, v], N[v, w], N[u, w] \neq \emptyset$, and $E(u, v, w) = \emptyset$ otherwise.
Now, we are ready to describe the following theorems, which characterize all the above mentioned

Helly classes, in terms of extensions $E(u,v,w)$ and universal sets $U[u,v,w]$. Theorem 2.3 is a reformulation of the results given in [5].

Theorem 2.1 *[1, 8, 6] A graph G is clique-Helly if and only if for every extension $E(u,v,w)$ such that u,v,w induce a triangle, there exists a vertex $z \in V(G)$ such that $E(u,v,w) \subseteq N[z]$, i.e. $U[u,v,w] \neq \emptyset$.*

Theorem 2.2 *A graph G is closed neighborhood-Helly if and only if for every extension $E(u,v,w)$, there exists a vertex $z \in V(G)$ such that $E(u,v,w) \subseteq N[z]$, i.e. $U[u,v,w] \neq \emptyset$.*

Theorem 2.3 *[5] A graph G is open neighborhood-Helly if and only if does not contain triangles and for every extension $E(u,v,w)$ of vertices u,v,w that induce a co-triangle, there exists a vertex $z \in V(G)$ such that $E(u,v,w) \subseteq N[z]$, i.e. $U[u,v,w] \neq \emptyset$.*

Theorem 2.4 *[7, 6] A graph is hereditary clique-Helly graph if and only if for every extension $E(u,v,w)$ such that u,v,w induce a triangle, there exists a vertex $z \in \{u,v,w\}$ such that $E(u,v,w) \subseteq N[z]$, i.e. $U[u,v,w] \cap \{u,v,w\} \neq \emptyset$ or $E(u,v,w) = \emptyset$.*

Theorem 2.5 *A graph is hereditary closed neighborhood-Helly graph if and only if for every extension $E(u,v,w)$ such that u,v,w induce a triangle, there exists a vertex $z \in \{u,v,w\}$ such that $E(u,v,w) \subseteq N[z]$, i.e. $U[u,v,w] \cap \{u,v,w\} \neq \emptyset$ or $E(u,v,w) = \emptyset$.*

Theorem 2.6 *A graph is hereditary open neighborhood-Helly graph if and only if it does not contain triangles and for every extension $E(u,v,w)$ of vertices u,v,w that induce a co-triangle, there exists a vertex $z \in \{u,v,w\}$ such that $E(u,v,w) \subseteq N[z]$, i.e. $U[u,v,w] \cap \{u,v,w\} \neq \emptyset$ or $E(u,v,w) = \emptyset$.*

The formulations of these characterizations clearly show the relations between all classes.
Finally, these characterizations allow new recognition algorithms and their complexities are summarized in Table 1 (most of them improve time complexities of previous algorithms).

3 MATRIX CHARACTERIZATIONS

We also characterize hereditary clique-Helly (hereditary closed neighborhood-Helly, hereditary open neighborhood-Helly) graphs from another point of view. We characterize them by forbidden submatrices of the incidence matrix of the graph.
We need some additional definitions.
Let $C_1, \ldots, C_k$ and $v_1, \ldots, v_n$ be the cliques and vertices of a graph G, respectively. We define M a *clique matrix* of G, as a 0-1 matrix whose entry (i,j) is 1 if $v_j \in C_i$ and 0 otherwise. The *augmented adjacency matrix* A^* of G is the one obtained from the adjacency matrix A of G, by setting to 1 each entry of its main diagonal. A permutation matrix is a matrix obtained by permuting the rows of an $n \times n$ identity matrix according to some permutation of the numbers 1 to n. Every row and column therefore contains precisely a single 1 with 0s everywhere else, and every permutation corresponds to a unique permutation matrix. A co-permutation matrix is one obtained from a permutation matrix, by replacing each 0 by 1 and each 1 by 0.
The following theorems show a relationship between the hereditary Helly property and co-permutation matrices.

Theorem 3.1 *[9] A graph G is hereditary clique-Helly if and only if its clique matrix M does not contain any co-permutation 3×3 submatrix.*

Theorem 3.2 *A graph G is hereditary closed neighborhood-Helly if and only if its augmented adjacency matrix A^* does not contain any co-permutation 3×3 submatrix.*

Theorem 3.3 *A graph G is hereditary open neighborhood-Helly if and only if its adjacency matrix A does not contain any co-permutation 3×3 submatrix.*

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Table 1

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Faster recognition of clique-Helly and hereditary clique-Helly graphs

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Abstract

A family of subsets of a set is *Helly* when every subfamily of it, which is formed by pairwise intersecting subsets contains a common element. A graph G is *clique-Helly* when the family of its (maximal) cliques is Helly, while G is *hereditary clique-Helly* when every induced subgraph of it is clique-Helly. The best algorithms currently known to recognize clique-Helly and hereditary clique-Helly graphs have complexities $O(nm^2)$ and $O(n^2m)$, respectively, for a graph with n vertices and m edges. In this Note, we describe algorithms which recognize both classes in $O(m^2)$ time. These algorithms also reduce the complexity of recognizing some other classes, as disk-Helly graphs.

Keywords: Algorithms; Clique-Helly graphs; Disk-Helly graphs; Helly property; Hereditary clique-Helly graphs; Hereditary disk-Helly graphs

1. Introduction

The Helly property has been studied in many contexts, as in combinatorics and geometry. Within graph theory, the Helly property has been applied to some different families of sets. Its application to the (maximal) cliques of a graph is one of the most common. It has lead to the classes of clique-Helly and hereditary clique-Helly graphs. Clique-Helly graphs have been considered in many papers, [4,15,10,11,9], among others. Hereditary clique-Helly graphs have been studied in, e.g., [13,5]. Clique-Helly graphs have been characterized in [8,16], while [13,17] contain characterizations for hereditary clique-Helly graphs. These characterizations lead to recognition algorithms whose complexities are $O(nm^2)$ for clique-Helly and $O(n^2m)$ for hereditary clique-Helly graphs, where n and m are the number of vertices and edges of the graph (see [6]). In this Note, we describe algorithms which reduce the complexities to $O(m^2)$, in both cases. We remark that all mentioned algorithms are based on Berge's basic test for Helly hypergraphs [3]. Finally, we mention that hereditary Helly hypergraphs can be also recognized in polynomial time [7].

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Let G be an undirected graph with vertex set $V(G)$ and edge set $E(G)$. Write ab to denote the edge of G, formed by vertices $a, b \in V(G)$. Represent by $N(a)$ the subset of vertices adjacent to a, and let $N[a] = N(a) \cup \{a\}$. Write $d_a = |N(a)|$. For an edge $ab \in E(G)$, define $N(ab) = N(a) \cap N(b)$ and $N[ab] = N[a] \cap N[b]$. Clearly, $N[ab] = N(ab) \cup \{a, b\}$. A vertex $a \in V(G)$ satisfying $N[a] = V(G)$ is a *universal* vertex of G. For $V' \subseteq V(G)$, denote by $G[V']$ the subgraph of G induced by the vertices of V'. Denote by $U(ab)$ the set of universal vertices of $G[N(ab)]$. Similarly, $U[ab]$ is the set of universal vertices of $G[N[ab]]$. Again, $U[ab] = U(ab) \cup \{a, b\}$.

Say that a family of subsets of some set is *Helly* when every subfamily of it, which is formed by pairwise intersecting subsets, contains a common element. A *clique* of a graph G is a maximal subset of pairwise adjacent vertices. Say that G is *clique-Helly* when the family of cliques of G form a Helly family, while G is *hereditary clique-Helly* when every induced subgraph of G is clique-Helly.

Let $T \subseteq V(G)$ be a subset of three vertices, forming a triangle. Denote by $T^* \subseteq V(G)$ the subset formed by the vertices of G which are adjacent to at least two vertices of T. Clique-Helly graphs have been characterized in terms of T^*, as follows.

Proposition 1. *(See [8,16].) A graph G is clique-Helly if and only if every triangle T of G is such that T^* has a universal vertex.*

Hereditary clique-Helly graphs have been characterized by forbidden subgraphs, as below.

Proposition 2. *(See [13,17].) A graph is hereditary clique-Helly if and only if it does not contain any of the graphs of Fig.* 1, *as induced subgraphs.*

The recognition algorithms for these classes are based on the above characterizations.

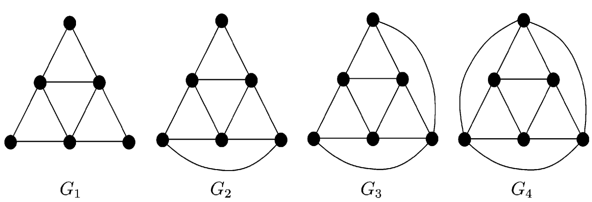

Fig. 1. Forbidden induced subgraphs of hereditary clique Helly graphs.

2. The algorithms

In this section, we describe the new implementations for recognizing clique-Helly and hereditary clique-Helly graphs. The former class is considered first. The method is based on the following proposition.

Proposition 3. *A graph G is clique-Helly if and only if $U[ab] \cap U[bc] \cap U[ac] \neq \emptyset$, for any triple of vertices $a, b, c \in V(G)$ forming a triangle.*

The proof follows directly from Proposition 1. The recognition algorithm basically checks the above conditions for every triangle of the graph.

Algorithm 1 *(Clique-Helly graphs).* Let G be the input graph. In the *initial step*, compute the sets $N(ab)$, $N[ab]$ and $U[ab]$, for every edge $ab \in E(G)$. In the *general step*, perform the following operations, for each edge $ab \in E(G)$.

For each $c \in N(ab)$, compute $S_c := U[bc] \cap U[ac]$. If every $w \in S_c$ satisfies $w \notin U[ab]$ then report "G is not clique-Helly" and *stop*.

Report "G is clique-Helly" and *stop*.

Clearly, for recognizing G as a clique-Helly graph, the algorithm checks whether $U[ab] \cap U[bc] \cap U[ac] \neq \emptyset$, for every triangle a, b, c of G. Consequently, the algorithm is correct.

Next, we evaluate the complexity of the algorithm. The sets $N(ab)$, $N[ab]$ and $U[ab]$ can be computed in overall $O(m^2)$ time, for all edges of G, with no difficulty. The operations performed within the general step of the algorithm depend on the size of the set $S_c = U[bc] \cap U[ac]$. Clearly, $|U[bc]| \leqslant |N[c]| = d_c + 1$, and $|U[ac]| \leqslant |N[c]| = d_c + 1$. Consequently, we need no more than $O(m)$ operations, in order to compute all sets S_c, for each edge $ab \in E(G)$. In addition, the sum of the sizes of the sets S_c is also $O(m)$, for each considered edge ab. To verify whether $w \notin U[ab]$ can be done in constant time, by employing boolean vectors. Consequently, we require $O(m)$ steps for each edge ab. That is, the general step of the algorithm also requires $O(m^2)$ time, which is the overall complexity.

In the sequel, we consider hereditary clique-Helly graphs. The proposed algorithm is based on the following proposition.

Proposition 4. *A graph G is hereditary clique-Helly if and only if $c \in U(ab)$, or $b \in U(ac)$, or $a \in U(bc)$, for each triangle with vertices $a, b, c \in V(G)$.*

Proof. Clearly, $c \in U(ab)$ if and only if $N(ab) \subseteq N(c)$. Similar relations hold for $b \in U(ac)$ and $a \in U(bc)$. Consequently, $c \in U(ab)$, or $b \in U(ac)$, or $a \in U(bc)$ holds if and only if $N(ab) \subseteq N(c)$, or $N(bc) \subseteq N(a)$, or $N(ac) \subseteq N(b)$.

Suppose G is hereditary clique-Helly. By Proposition 2, G does not contain any of the graphs of Fig. 1 as induced subgraphs. Consequently, for any triangle with vertices a, b, c, there are no vertices $x, y, z \in V(G)$ which simultaneously satisfy $x \in N(ab) \setminus N(c)$, $y \in N(bc) \setminus N(a)$ and $z \in N(ac) \setminus N(b)$. Consequently, $N(ab) \subseteq N(c)$, or $N(bc) \subseteq N(a)$, or $N(ac) \subseteq N(b)$, meaning that the proposition is true. The proof of the converse is similar. □

The algorithm for recognizing hereditary clique-Helly graphs is a direct implementation of Proposition 4.

Algorithm 2 *(Hereditary clique-Helly graphs).* Let G be the input (connected) graph. In the *initial step*, compute $U(ab)$, for each edge $ab \in E(G)$. In the *general step*, for each triangle a, b, c of G, perform the following operations.

If $c \notin U(ab)$ and $b \notin U(ac)$ and $a \notin U(bc)$ then report "G is not hereditary clique-Helly" and *stop*.

Report "G is hereditary clique-Helly" and *stop*.

As for the complexity, the initial step requires $O(m^2)$ time. In the general step, to verify if $c \notin U(ab)$ can be done in constant time, again employing boolean vectors. Similarly, for the checks $b \notin U(ac)$ and $a \notin U(bc)$. Consequently, the total complexity of the general step is that of the number of triangles of G, that is, $O(nm)$. Therefore the overall complexity of the algorithm is $O(m^2)$.

3. Conclusions

We have described new implementations for performing the required checks on the triangles of a graph G, which leads to recognizing whether G is clique-Helly or hereditary clique-Helly. The complexity of the proposed methods is $O(m^2)$, for any of these two classes. Currently known algorithms recognize clique-Helly graphs in $O(nm^2)$ time, and hereditary clique-Helly in $O(n^2m)$ time.

Besides improving the complexity of recognizing these classes, the proposed algorithms also represent improvements in the recognition of some other classes. Disk-Helly graphs and hereditary disk-Helly graphs are examples of such classes.

For disk-Helly graphs, the currently best recognition algorithm has complexity $O(n^2m)$ [1]. On the other hand, in [2] it has been proved that a graph is disk-Helly if and only if it is dismantlable and clique-Helly. Dismantlable graphs can be recognized in $O(nm)$ time [12, 14]. Consequently, by applying Algorithm 1, we can reduce to $O(m^2)$ the complexity of recognizing disk-Helly graphs.

For hereditary disk-Helly graphs, the recognition is based on the characterization [8] (cf. [6]), which states that a graph G is hereditary disk-Helly if and only if G is chordal and does not contain the graph G_1 as an induced subgraph. Since chordal graphs can be recognized in linear time, the complexity is dominated by the operations of verifying if G contains G_1 as an induced subgraphs, which would require $O(n^2m)$ time. However, since the graphs G_2, G_3 and G_4 are not chordal, we can apply Algorithm 2 after the chordality test, and therefore reduce the total complexity to $O(m^2)$ time.

We leave the question whether clique-Helly graphs or hereditary clique-Helly graphs can be recognized in time proportional to the number of triangles of the graph.

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